Mathematics 1	Handouts: Week 3

Kronecker product of matrices

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1. Kronecker (tensor) product

The Kronecker product (also a *tensor product*) of matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is a $mp \times nq$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}.$$

If A, B, C and D are matrices of appropriate sizes, then

(1) $0 \otimes A = A \otimes 0 = 0$
(2) $\alpha \otimes A = A \otimes \alpha = \alpha A$ for any $\alpha \in \mathbb{R}$
(3) $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha(A \otimes B)$
(4) $(A + B) \otimes C = A \otimes C + B \otimes C$ and $A \otimes (B + C) = A \otimes B + A \otimes C$
(5) $(A \otimes B)^{\mathsf{T}} = A^{\mathsf{T}} \otimes B^{\mathsf{T}}$
(6) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.
(7) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
(8) $\|A \otimes B\|_F = \|A\|_F \|B\|_F$.
(9) If $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
(10) If $A \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \dots, \lambda_m$ and B has eigenvalues $\mu_1, \dots, \mu_n$, then the set of eigenvalues of $A \otimes B$ is equal to $\{\lambda_i \mu_j\}$.
(11) If $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$, then $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \operatorname{tr}(B)$.
(12) If $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$, then $\det(A \otimes B) = (\det A)^m (\det B)^n$.
(13) $\operatorname{rank}(A \otimes B) = \operatorname{rank}(A) \operatorname{rank}(B)$
(14) If $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{p\times r}$, then

$$\operatorname{vec}(ABC) = (C^{\mathsf{T}} \otimes A) \operatorname{vec}(B).$$

2. Further reading

(1) David A. Harville: Matrix Algebra From a Statistian's Perspective, Springer, 1997, Sections 16.1., 16.2. and 16.3.
(2) Roger A. Horn and Charles R. Johnson, *Topics in Matrix Analysis*, Cambridge, 2006, Section 4.2.
(3) Charles F. Van Loan:The ubiquitous Kronecker product, Journal of Computational and Applied Mathematics 123 (2000) 85-100.

3. Homework

(1) Solve the quiz on Učilnica.
(2) David A. Harville: Matrix Algebra From a Statistian's Perspective, Springer, 1997, page 368 (Exercises 1, 2, 3, 4, 14, 15, 16).