Selected topics

- Digital filters
- Digital filters and their properties
- Marginal stability
- FIR filters implemented as IIR filters, integer multiplier filters
- Cascading filters
- Time reversal filtering
- Computational complexity of the DFT

Digital filters

- Arrange the properties according to type of filter

FIR filters | IIR filters | Property

Conceptually limited (zeros only)
Conceptually wider (zeros, poles)
Feedback
No feedback
Always stable
May be unstable
Can be linear phase
Nonlinear phase (could be close)
Convolution computation possible
Filter order (4 – 20)
Filter order (20 – 2000)
Computationally more expensive
Computationally less expensive
Could be derived from analog prototypes
Unrelated to continuous time filtering

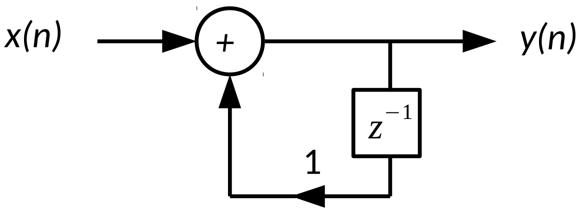
Digital filters and their properties

- Properties

FIR filters	IIR filters
- Conceptually limited (zeros only)	+ Conceptually wider (zeros, poles)
No feedback	Feedback
+ Always stable	- May be unstable
+ Can be linear phase	- Nonlinear phase (could be close)
+ Convolution computation possible	
- Filter order (20 – 2000)	+ Filter order (4 – 20)
- Computationally more expensive	+ Computationally less expensive
Unrelated to continuous time filtering	+ *Could be derived from analog prototypes*

Marginal stability

- An LTI system with poles on the unit circle is not stable (marginally stable)
- It produces an unbounded response when excited by an input signal that also has a pole at the same position on the unit circle
- Example, determine the step response of the following causal system



$$y(n) = y(n-1) + x(n)$$

The transfer function $H(z) = \dfrac{1}{1 - z^{-1}}$ contains a pole at $z = 1$ (unit circle)

Input signal, $x(n) = u(n)$ unit step signal

The Z transform of $x(n)$ $X(z) = \dfrac{1}{1 - z^{-1}}$ also contains pole at $z = 1$

Since $Y(z) = H(z)\,X(z) = \dfrac{1}{(1 - z^{-1})^2}$ a double pole at $z = 1$

The inverse Z transform $y(n) = (n+1)\,u(n)$ which is a ramp sequence

FIR filters implemented as IIR filters, integer multiplier filters

- Example, recall moving average

$M = 8$

$$y(n) = \frac{1}{M}\sum_{l=0}^{M-1} x(n-l) \qquad h(n)=\begin{cases}\frac{1}{M}, & 0 \le n \le M-1\\ 0, & \text{otherwise}\end{cases}$$

- The transfer function is

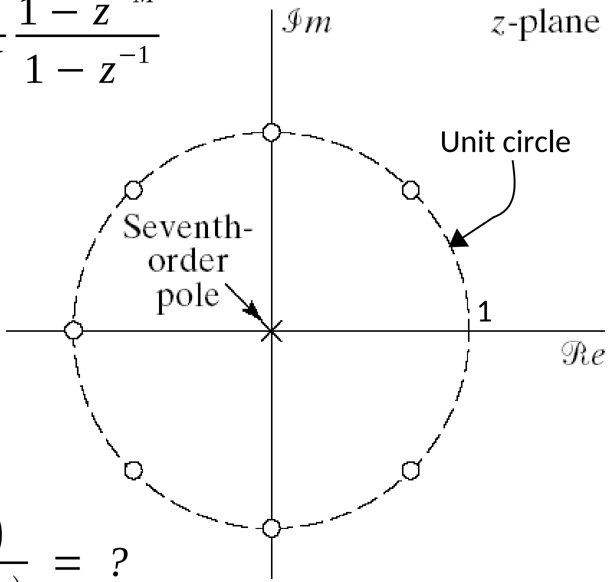
$$H(z)=\sum_{n=-\infty}^{\infty} h(n)z^{-n} = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1}{M}(1 + z^{-1} + \ldots + z^{-M+1}) = \frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}}$$

- The zeros, $z_{(k+1)}$, can be written as

$$z_{(k+1)} = a e^{j2\pi k/M}, \quad k = 0,1,\ldots,M-1$$

- For $k = 0$ we have a zero at $z_1 = 1$
- The zero cancels the pole at $p_1 = 1$

$$H(z) = \frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}} = \frac{1}{M}\frac{z^M}{z^M}\frac{(1-z^{-M})}{(1-z^{-1})} = \frac{1}{M}\frac{(z^M-1)}{z^{M-1}(z-1)} = ?$$



[Oppenheim, Schafer]

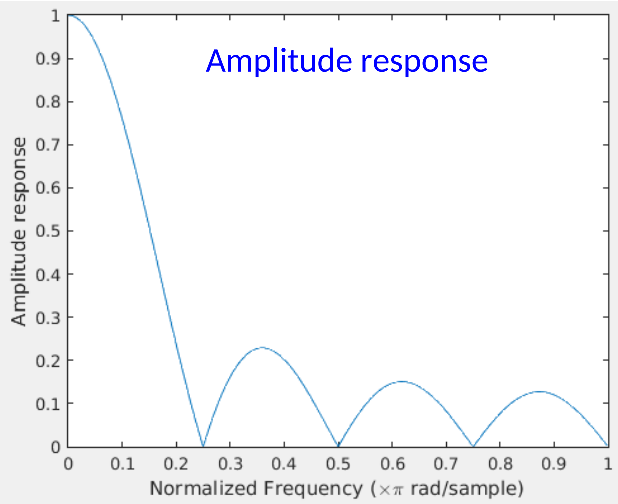
FIR filters implemented as IIR filters, integer multiplier filters

- Example, recall moving average

M = 8

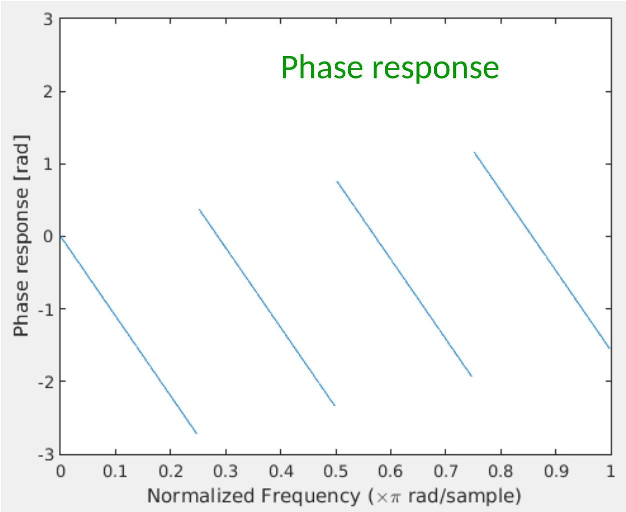
$$H(\omega) = |H(\omega)| . e^{j\theta(\omega)}$$

$$H(\omega) = \frac{1}{M} e^{-j\omega\frac{(M-1)}{2}} \frac{\sin(M\omega/2)}{\sin(\omega/2)}$$



$$|H(\omega)| = \left|\frac{1}{M}\right| \left|\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right|$$

$$\theta(\omega) = -\frac{(M-1)}{2}\omega + \pi r$$



- The output

$$y(n) = \frac{1}{M}\sum_{l=0}^{M-1} x(n-l)$$

$$y(n) = y(n-1) + x(n) - x(n-M) \qquad y'(n) \leftarrow \frac{1}{M} y(n)$$

FIR filters implemented as IIR filters, integer multiplier filters

- By proper selection of parameters *m* and *n*, poles on the unit circle are canceled
- FIR filters implemented as IIR filters → much less computation

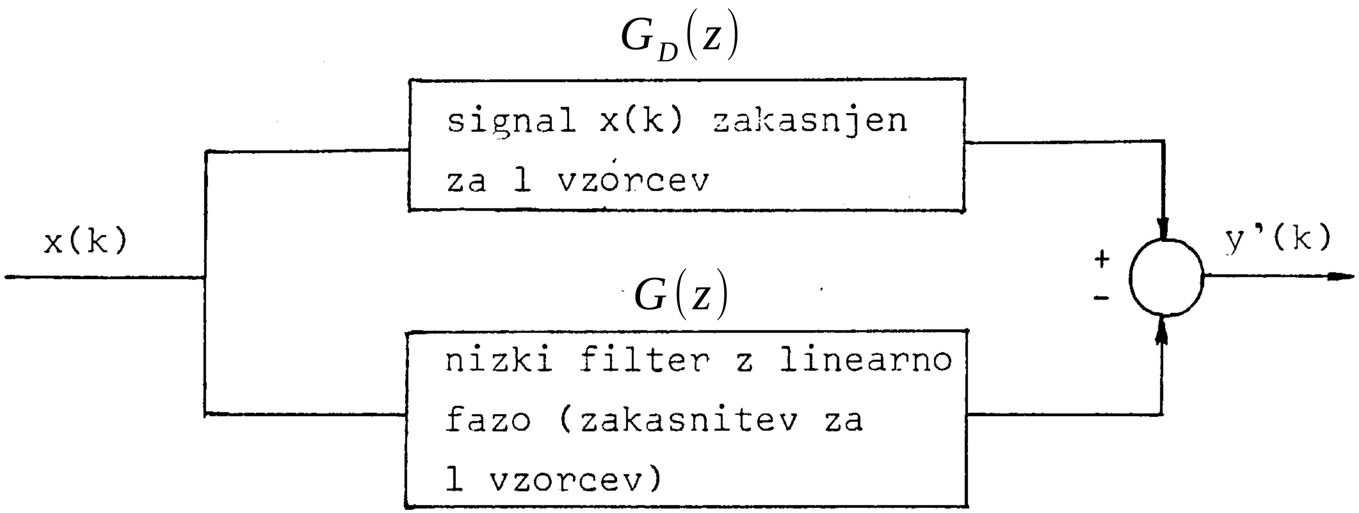
$$H(z) = \frac{(1 - z^{-m})^M}{(1 - z^{-1})^M}$$

$$H(z) = \frac{(1 - z^{-m})^M}{(1 - z^{-1})^M} \Big|_{(z = e^{j\omega})} = e^{-j\omega(\frac{m}{2} - \frac{1}{2})M} . \left(\frac{\sin(m/2\,\omega)}{\sin(1/2\,\omega)}\right)^M$$

$$H(z) = \frac{(1 + a_k z^{-m})^M}{(1 + b_k z^{-n})^M} \qquad a_k, b_k \in \mathbb{Z}$$

FIR filters implemented as IIR filters, integer multiplier filters

- How to obtain band-stop characteristic using band-pass filter (or, vice versa)?



$$G(z) = \frac{(1 + a_k z^{-m})^M}{(1 + b_k z^{-n})^M}$$

$$H(z) = G_D(z) - G(z) = \left(\frac{m}{n}\right)^M . z^{-j\omega\left(\frac{m}{2}.M - \frac{n}{2}.M\right)} - G(z)$$

FIR filters implemented as IIR filters, integer multiplier filters

- High-pass filtering using low-pass filter, *$H_{LP}(z)$*

$$H_{LP}(z) = \frac{\left(1 - z^{-344}\right)^2}{\left(1 - z^{-1}\right)^2} \qquad G(z) = \frac{\left(1 + a_k z^{-m}\right)^M}{\left(1 + b_k z^{-n}\right)^M}$$

ak = -1, bk = -1, m = 344, n = 1 in M = 2

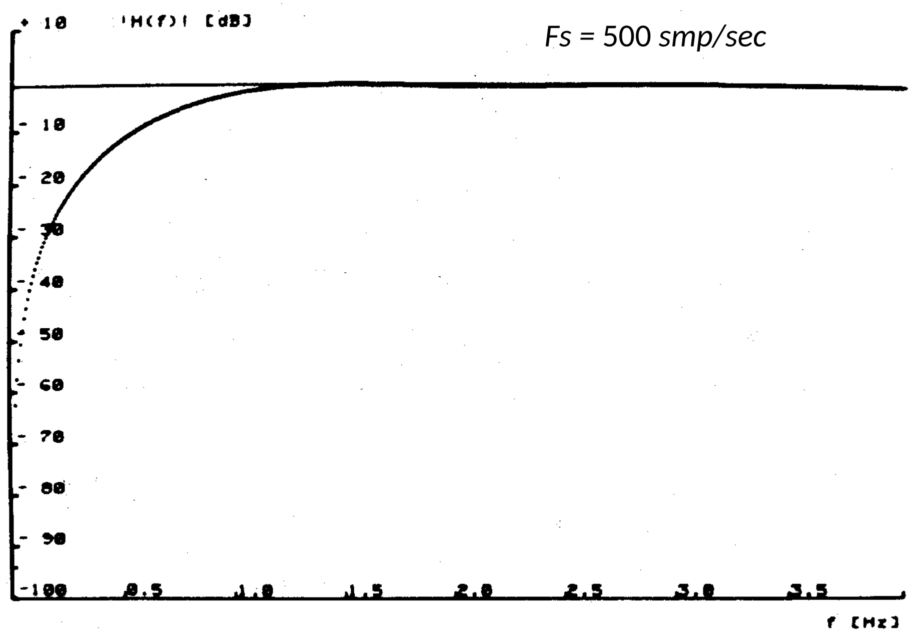
y(k) = 2·y(k - 1) - y(k - 2) + x(k) - 2·x(k - 344) +

+ x(k - 688)

y'(k) = k_v·x(k - 343) - y(k)

y'(k) ← y'(k)/k_v

$k_v = 344^2$



FIR filters implemented as IIR filters, integer multiplier filters

- High-pass and 50, 100, 150, 200, 250 Hz notch filtering using combined band-pass filter, i.e., low-pass and 50, 100, 150, 200, 250 Hz band-pass filter, $H_{L,50}(z)$

$$G(z) = \frac{\left(1 + a_k z^{-m}\right)^M}{\left(1 + b_k z^{-n}\right)^M}$$

$$H_{L,50}(z) = \frac{\left(1 - z^{-330}\right)^2}{\left(1 - z^{-10}\right)^2}$$

ak = -1, bk = -1, m = 330, n = 10 in M = 2

y(k) = 2·y(k − 10) − y(k − 20) + x(k) − 2·x(k − 330) +

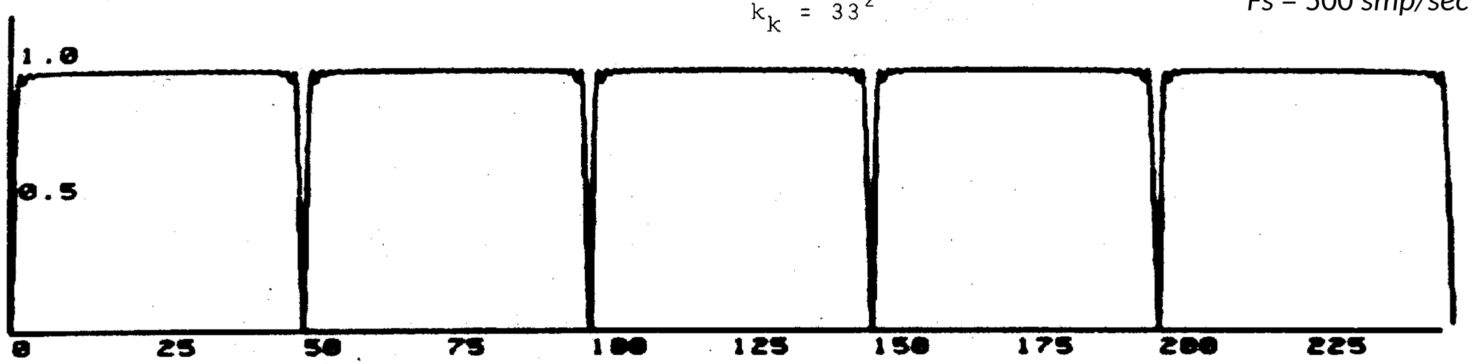
+ x(k − 660)

$y'(k) = k_k\, x(k - 320) - y(k)$

$y'(k) \leftarrow y'(k)/k_k$

$k_k = 33^2$

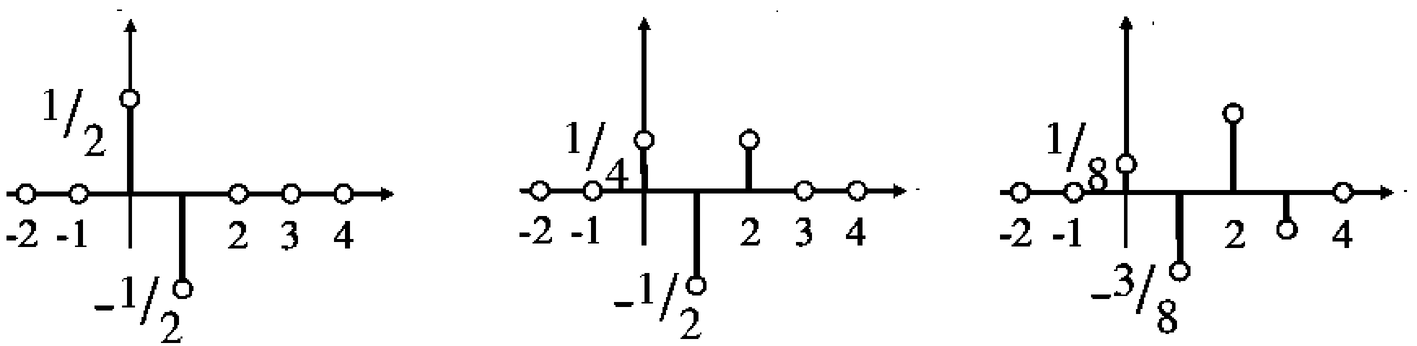
Fs = 500 *smp/sec*



Cascading filters

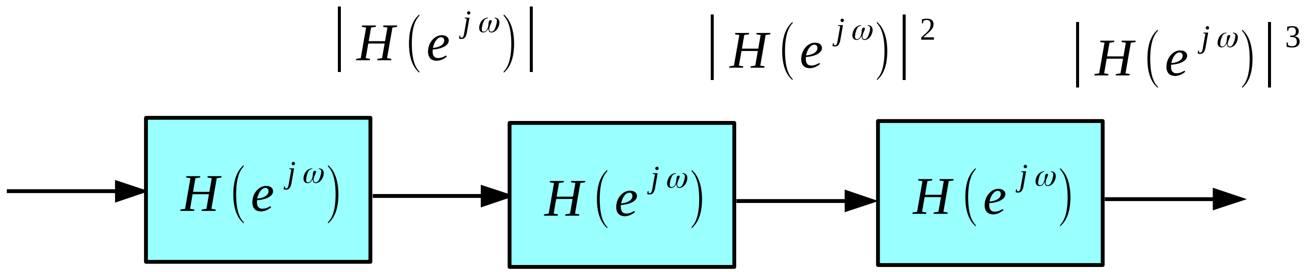
- Repeating a filter in cascade connection
  - Longer impulse response (for FIR)
  - $h_1(n) = h(n) = \{½, - ½\}$
  - $h_2(n) = h(n) * h(n) = \{¼, - ½, ¼\}$
  - $h_3(n) = h(n) * h(n) * h(n) = \{1/8, -3/8, 3/8, -1/8\}$

$h_1(n)$ $h_2(n)$ $h_3(n)$

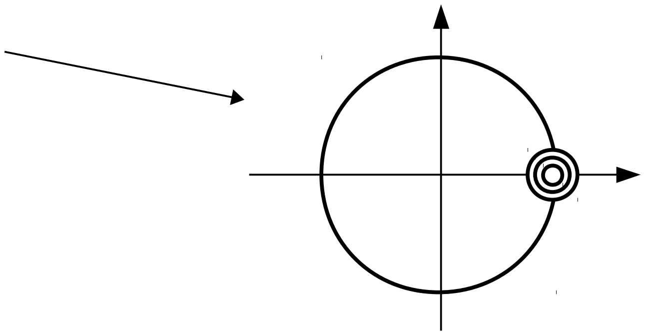


Cascading filters

- Repeating a filter in cascade connection
  - Transfer characteristic is becoming more abrupt
  - In general, given order, cascade filters will not be optimal

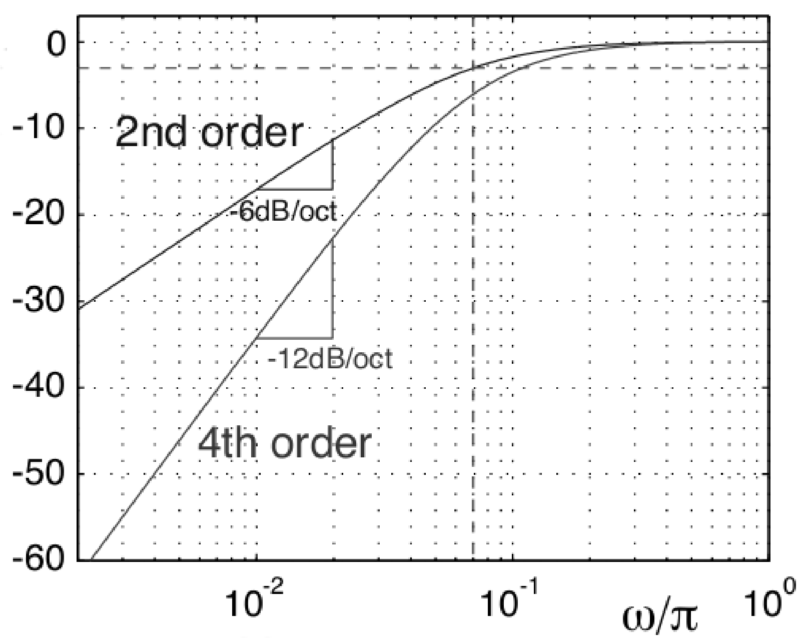


- Repeated zeros and poles in Z plane



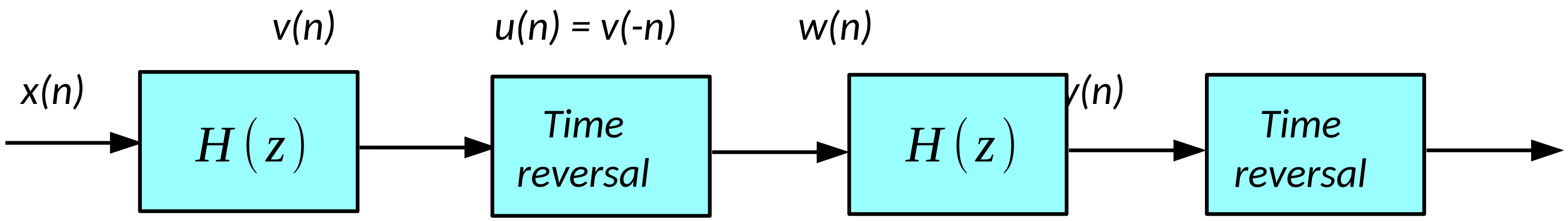
Cascading filters

- Repeating a filter in cascade connection
  - Roll-off slope improved
  - Gain at the cutoff frequency changes at the rate *N*.3dB (*N* - order)



Time reversal filtering

- Double-pass filtering scheme



→ *Zero phase result, off-line analysis, non-causal, needs entire signal*

$$v(n) = x(n) * h(n) \rightarrow V(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$u(n) = v(-n) \rightarrow U(e^{j\omega}) = V(e^{-j\omega}) = V^*(e^{j\omega})$$

$$w(n) = u(n) * h(n) \rightarrow W(e^{j\omega}) = H(e^{j\omega})U(e^{j\omega})$$

$$y(n) = w(-n) \rightarrow Y(e^{j\omega}) = W^*(e^{j\omega}) = (H(e^{j\omega})(H(e^{j\omega})X(e^{j\omega}))^*)^*$$

$$\rightarrow Y(e^{j\omega}) = X(e^{j\omega})\left|H(e^{j\omega})\right|^2$$

See also: https://www.youtube.com/watch?v=ue4ba_wXV6A

Computational complexity of the DFT

- Using basic DFT formula

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0,1,2,\dots,N-1 \qquad W_N = e^{-j\frac{2\pi}{N}}$$

  - Computational complexity of the order *O(N.N)*
    - * Each point requires *N* complex multiplications and *N* – 1 complex additions
    - * Therefore, *N* points, what yields *N.N* multiplications and *N.(N* – 1*)* additions
  - Example DFT (*N* = 1024)
    - * Complex multiplications, *N.N* = 1 048 576
    - * Complex additions, *N.(N* – 1*)* = 1 047 552
- *There is the Fast Fourier Transform (FFT) algorithm*
  - Computational complexity of the order *O(N.log(N))*