Transform domain analysis of discrete-time signals and systems, II

- Transfer function of LTI systems
- Frequency response for rational system functions
- Finite Impulse Response (FIR) systems
- Phase delay
- Finite Impulse Response (FIR) systems
- Infinite Impulse response (IIR) systems
- Group delay
- Infinite Impulse response (IIR) systems
- Stability
- Review of LTI systems – the Big Picture
- (Additional materials)

Transfer function of LTI systems

- Linear Constant-Coefficient Difference Equations (LCCDE) are
  - General to represent most useful systems, and implementable
  - Linear, time-invariant, and causal with zero initial conditions

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

- Transfer (system) function (Transfer characteristic) of LTI systems defined by LCCDE

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0}{a_0} z^{(N-M)} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

- *For causal systems (real a_k and b_k, or equivalently real h(n))*

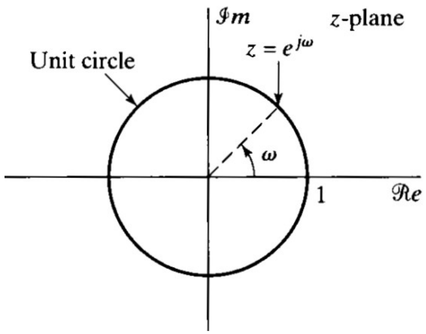
  1. The roots of the polynomials are either real or complex conjugate pairs

  2. The order of numerator cannot be greater than the order of denominator

Frequency response for rational system functions

- Transfer function, *H(z)*, (rational function) of LTI systems

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0}{a_0} z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$



$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

Frequency response: $$H(e^{j\omega}) = \frac{b_0}{a_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

$$H(e^{j\omega}) = |H(e^{j\omega})| \,.\, e^{j\theta(\omega)}$$

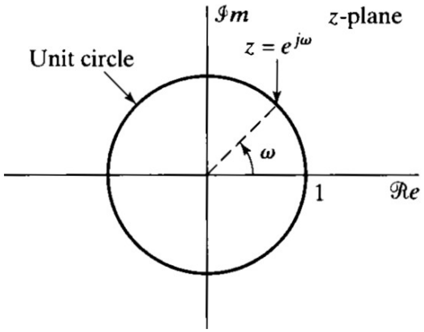
Amplitude response: $$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}|e^{j\omega} - z_k|}{\prod_{k=1}^{N}|e^{j\omega} - p_k|}, \qquad |e^{j\omega(N-M)}| = 1$$

[Oppenheim, Schafer]

Frequency response for rational system functions

- Transfer function, *H(z)*, (rational function) of LTI systems

Frequency response: $$H(e^{j\omega}) = \frac{b_0}{a_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega}-z_k)}{\prod_{k=1}^{N}(e^{j\omega}-p_k)}$$



$$H(e^{j\omega}) = |H(e^{j\omega})| . e^{j\theta(\omega)}$$

Phase response: $$\theta(\omega) = \arg\{\frac{b_0}{a_0}\} + \omega(N-M) + \sum_{k=1}^{M}\arg\{e^{j\omega}-z_k\} - \sum_{k=1}^{N}\arg\{e^{j\omega}-p_k\}$$

Operator arg (as opposed to $\sphericalangle$) means unwrapped phase

 [Oppenheim, Schafer]

Finite Impulse Response (FIR) systems

- The transfer function does not have any poles, except at $z = 0$
  - Finite Impulse Response (FIR) filters

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \sum_{k=0}^{M} b_k z^{-k} = b_0 z^{-M} \prod_{k=1}^{M} (z - z_k)$$

- The impulse response is of finite length

$$h(n) = \sum_{k=0}^{M} b_k \delta(n-k)$$

- Properties
  - Impulse response, $h(n)$, of finite duration, defined simply by b_k coefficients
  - Always stable (no feedback connection)
  - Linear phase (can have linear phase)
  - Larger number of coefficients needed
  - Implementation using convolution sum possible, $b_k = h(k)$, $k = 0, 1, 2, \dots, N$

LCCDE equation (FIR)

$$y(n) = \sum_{k=0}^{M} b_k \, x(n-k)$$

Finite Impulse Response (FIR) systems

- Example, recall moving average

$M = 8$

$$y(n) = \frac{1}{M}\sum_{l=0}^{M-1} x(n-l)$$

$$h(n) = \begin{cases} \frac{1}{M}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

- The transfer function is

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = 1 + z^{-1} + \ldots + z^{-M+1} = \frac{1}{M}\frac{1 - z^{-M}}{1 - z^{-1}}$$

$$H(e^{j\omega}) = \frac{1}{M} e^{-j\omega\frac{(M-1)}{2}} \frac{\sin(M\omega/2)}{\sin(\omega/2)} \longrightarrow H(e^{j\omega}) = |H(e^{j\omega})| \,.\, e^{j\theta(\omega)}$$

$$|H(e^{j\omega})| = \left|\frac{1}{M}\right| \left|\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right| \qquad \theta(\omega) = -\frac{(M-1)}{2}\omega + \pi r$$

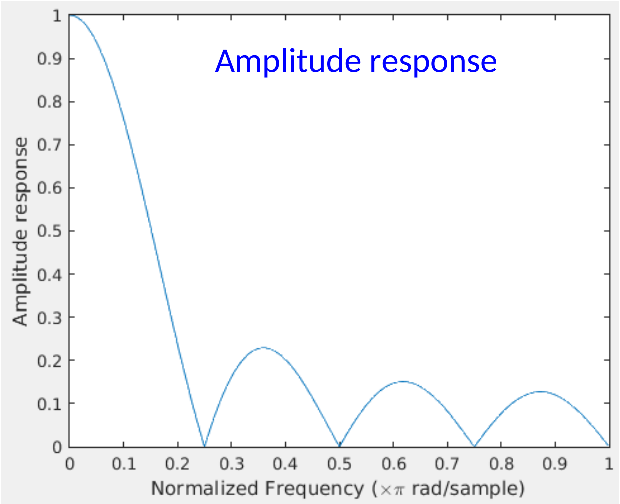
Finite Impulse Response (FIR) systems

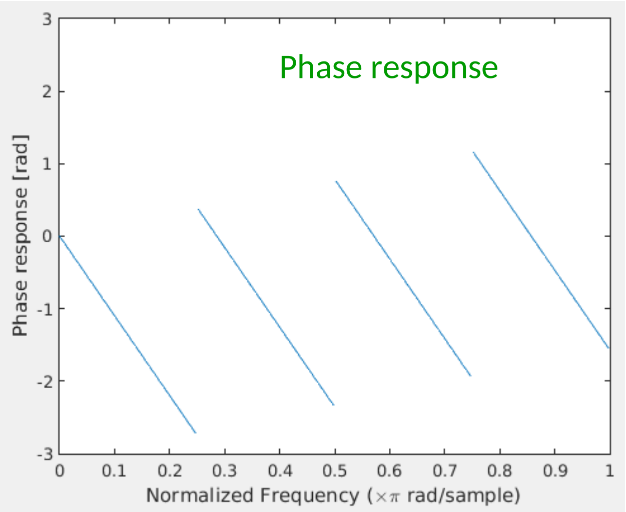
- Example, recall moving average

M = 8

$$H(e^{j\omega}) = |H(e^{j\omega})| . e^{j\theta(\omega)}$$

$$H(e^{j\omega}) = \frac{1}{M} e^{-j\omega\frac{(M-1)}{2}} \frac{\sin(M\omega/2)}{\sin(\omega/2)}$$





$$|H(e^{j\omega})| = \left|\frac{1}{M}\right| \left|\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right|$$

$$\theta(\omega) = -\frac{(M-1)}{2}\omega + \pi r$$

- The output, *y(n)*

$$H(z) = \frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}} \rightarrow y(n) = y(n-1) + \frac{1}{M}(x(n) - x(n-M))$$

Finite Impulse Response (FIR) systems

- Example, recall moving average

$M = 8$

$$y(n) = \frac{1}{M}\sum_{l=0}^{M-1} x(n-l)$$

$$h(n) = \begin{cases} \frac{1}{M}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

- The transfer function is

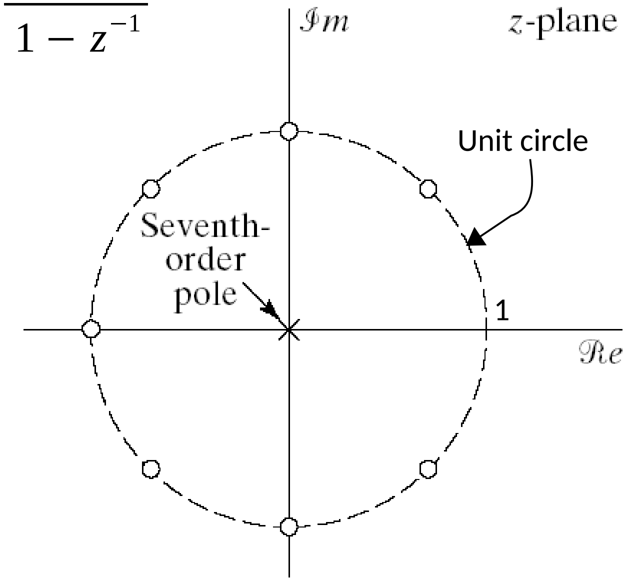
$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = 1 + z^{-1} + \dots + z^{-M+1} = \frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}}$$

$$H(z) = \frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}} = \frac{1}{M}\frac{z^M}{z^M}\frac{(1-z^{-M})}{(1-z^{-1})} = \frac{1}{M}\frac{(z^M-1)}{z^{M-1}(z-1)}$$

- The zeros, $z_{(k+1)}$, can be written as

$$z_{(k+1)} = e^{j2\pi k/M}, \quad k = 0,1,\dots,M-1$$

- There are M zeros
- For $k = 0$ we have a zero at $z_1 = 1$
- This zero cancels a pole at $p_1 = 1$



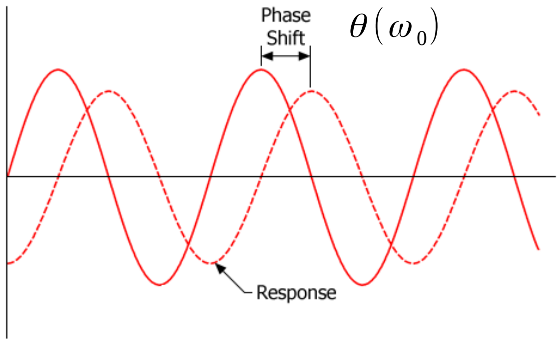
[Oppenheim, Schafer]

Phase delay

- Phase delay is negative phase response divided by frequency (i.e., phase response delay in samples at a frequency point)
  - The phase delay gives the time delay in seconds experienced by each sinusoidal component of the input signal

$$\tau_P(\omega) = \frac{-\theta(\omega)}{\omega}$$

- For sinusoidal signal



$$x(n) = A\cos(\omega_0 n + \Phi)$$

$$y(n) = A\left|H(e^{j\omega_0})\right|\cos(\omega_0 n + \theta(\omega_0) + \Phi)$$

$$y(n) = A\left|H(e^{j\omega_0})\right|\cos\left(\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right) + \Phi\right)$$

$$y(n) = A\left|H(e^{j\omega_0})\right|\cos(\omega_0(n - \tau_P(\omega_0)) + \Phi)$$

Phase shift (or time shift) In radians

Phase delay in samples

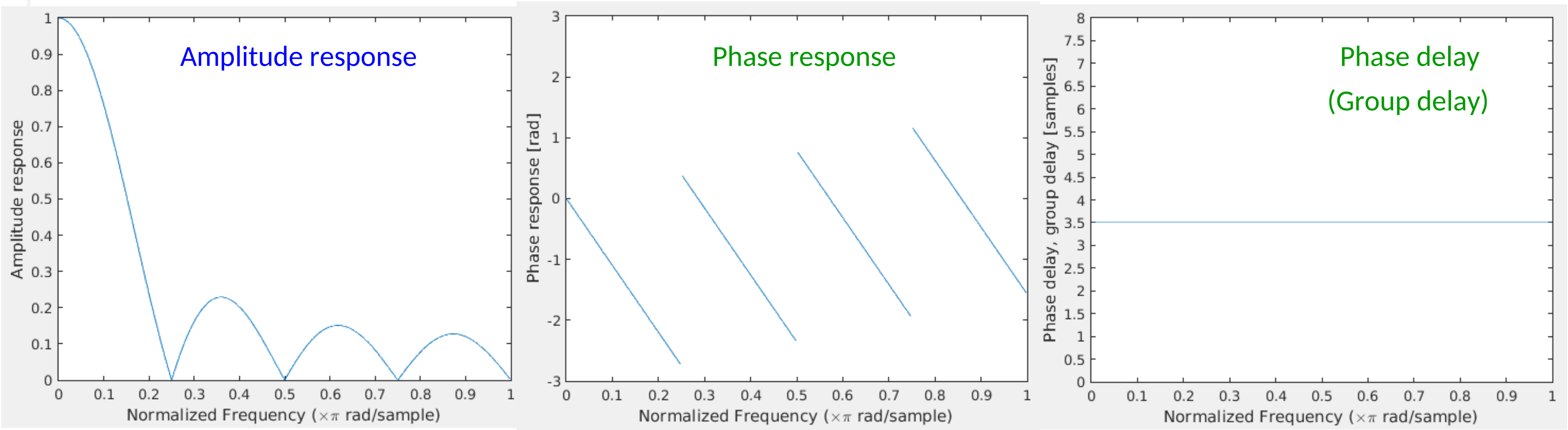
- Positive $\tau_P(\omega)$ means delay (causal)

Finite Impulse Response (FIR) systems

- Example, recall moving average

$M = 8$

$$H(e^{j\omega}) = \frac{1}{M} e^{-j\omega\frac{(M-1)}{2}} \frac{\sin(M\omega/2)}{\sin(\omega/2)}$$



$$|H(e^{j\omega})| = \left|\frac{1}{M}\right| \left|\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right|$$

$$\theta(\omega) = -\frac{(M-1)}{2}\omega + \pi r$$

$$\tau_P(\omega) = \frac{-\theta(\omega)}{\omega} = \frac{M-1}{2} = 3.5$$

$$\left(\tau_G(\omega) = \frac{-d\theta(\omega)}{d\omega} = \frac{M-1}{2} = 3.5\right)$$

- The output, *y(n)*, constant delay, *(M – 1)/2*, at all frequencies

Finite Impulse Response (FIR) systems

- Example of exam task

Derive the frequency, amplitude and phase response of causal moving average filter over three samples. Sketch the amplitude and phase response. Write the difference equation of the filter. What is the delay of the output signal in samples?

Infinite Impulse Response (IIR) systems

- At least one pole does not cancel with a zero
  - Infinite Impulse Response (IIR) filters

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0}{a_0} z^{(N-M)} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

- There is at least one term of the form $\frac{1}{1 - a z^{-1}} \leftrightarrow a^n u(n)$
- Properties
  - Impulse response, *h(n)*, of infinite duration, e.g., $y(n) = 0.5\, y(n-1) + x(n)$
  - Feedback connection
  - Can be unstable, e.g., $y(n) = 1.5\, y(n-1) + x(n)$
  - Non-linear phase (can be close to linear)
  - Lower number of coefficients needed

LCCDE equation (IIR)

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k)$$

Infinite Impulse Response (IIR) systems

- Example, “leaky integrator”, the first-order lowpass filter:

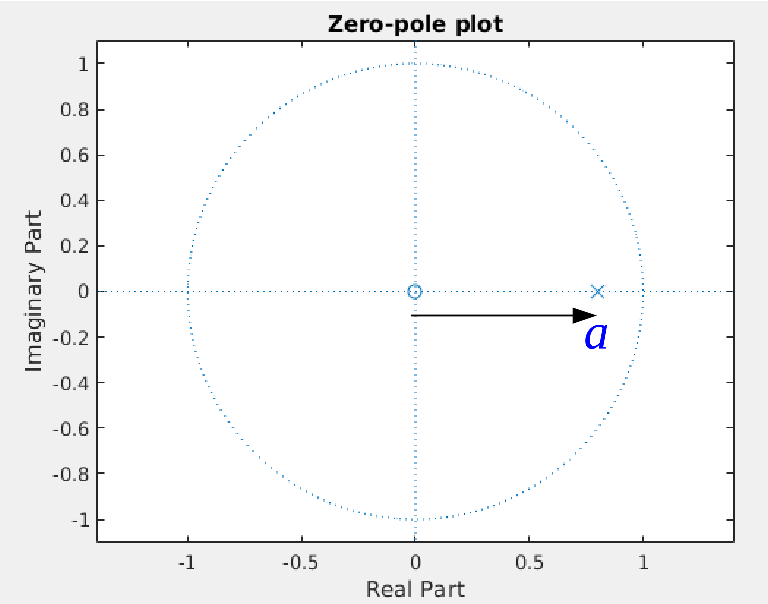
$$y(n) = a\,y(n-1) + x(n), \quad a = 0.8 \qquad y(n) = 0.8\,y(n-1) + x(n)$$

$$H(z) = \frac{1}{1 - a z^{-1}} = \frac{b_0}{a_0}\frac{z}{z}\frac{1}{(1 - 0.8 z^{-1})} = \frac{b_0}{a_0}\frac{z}{z - 0.8}, \qquad z_1 = 0,\ \ p_1 = 0.8,\ \ a_0 = 1$$

- Transfer function, zero-pole diagram

$$H(z) = \frac{b_0}{1 - a z^{-1}}$$

$$H(z) = \frac{b_0}{1 - 0.8 z^{-1}}$$



Infinite Impulse Response (IIR) systems

- Example, “leaky integrator”, the first-order lowpass filter:

$$y(n) = a\,y(n-1) + x(n), \quad a = 0.8 \qquad y(n) = 0.8\,y(n-1) + x(n)$$

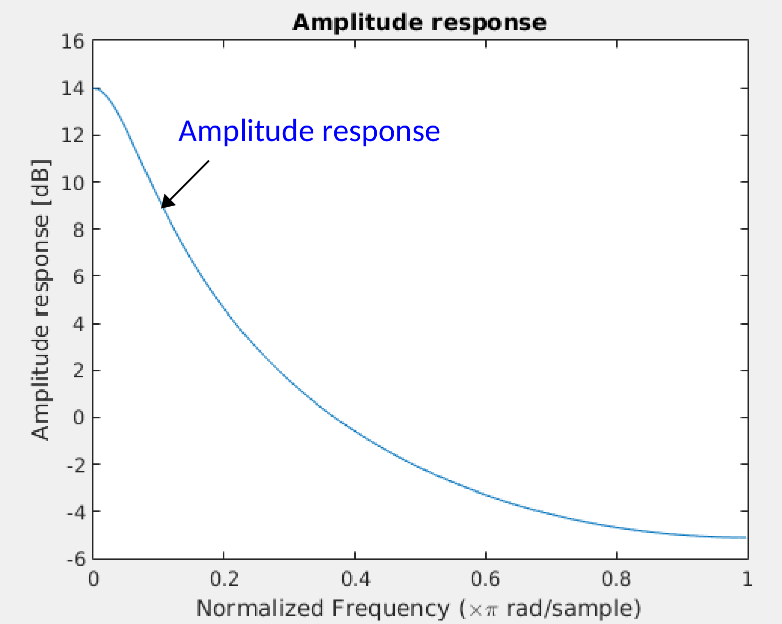
$$H(z) = \frac{1}{1 - a z^{-1}} = \frac{b_0}{a_0}\frac{z}{z}\frac{1}{(1 - 0.8 z^{-1})} = \frac{b_0}{a_0}\frac{z}{z - 0.8}, \qquad z_1 = 0, \ p_1 = 0.8, \ a_0 = 1$$

- Frequency response

$$H(e^{j\omega}) = \frac{b_0\, e^{j\omega}}{(e^{j\omega} - a)}$$

- Amplitude response $\quad (b_0 = 1)$

$$|H(e^{j\omega})| = \frac{b_0\,|e^{j\omega}|}{|e^{j\omega} - a|}$$



Infinite Impulse Response (IIR) systems

- Example, “leaky integrator”, the first-order lowpass filter:
- Frequency response

$$H\left(e^{j\omega}\right) = \frac{b_0\, e^{j\omega}}{\left(e^{j\omega} - a\right)} \qquad a = 0.8$$

- Amplitude response

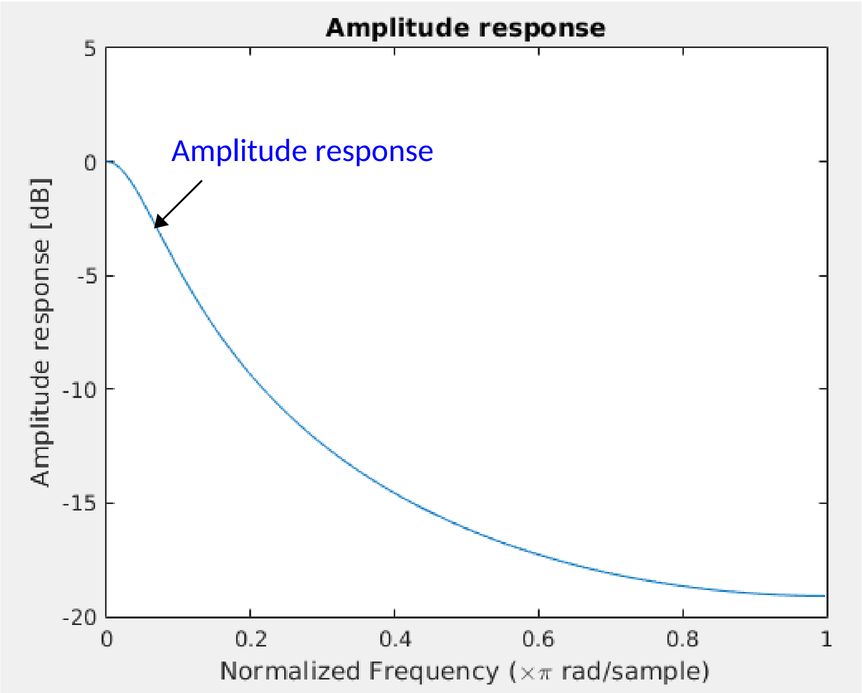
$$\left|H\left(e^{j\omega}\right)\right| = \frac{b_0 \left|e^{j\omega}\right|}{\left|e^{j\omega} - a\right|}$$

- Normalization factor, b_0

$$\left|H\left(e^{j\omega}\right)\right| = \frac{b_0 \left|e^{j\omega}\right|}{\left|e^{j\omega} - a\right|} = 1$$

$$\left|H\left(e^{j\omega}\right)\right| = \frac{b_0}{\left|1 - a\right|} = \frac{b_0}{\left|1 - 0.8\right|} = 1$$

$$b_0 = \left|1 - 0.8\right| = 0.2$$



Infinite Impulse Response (IIR) systems

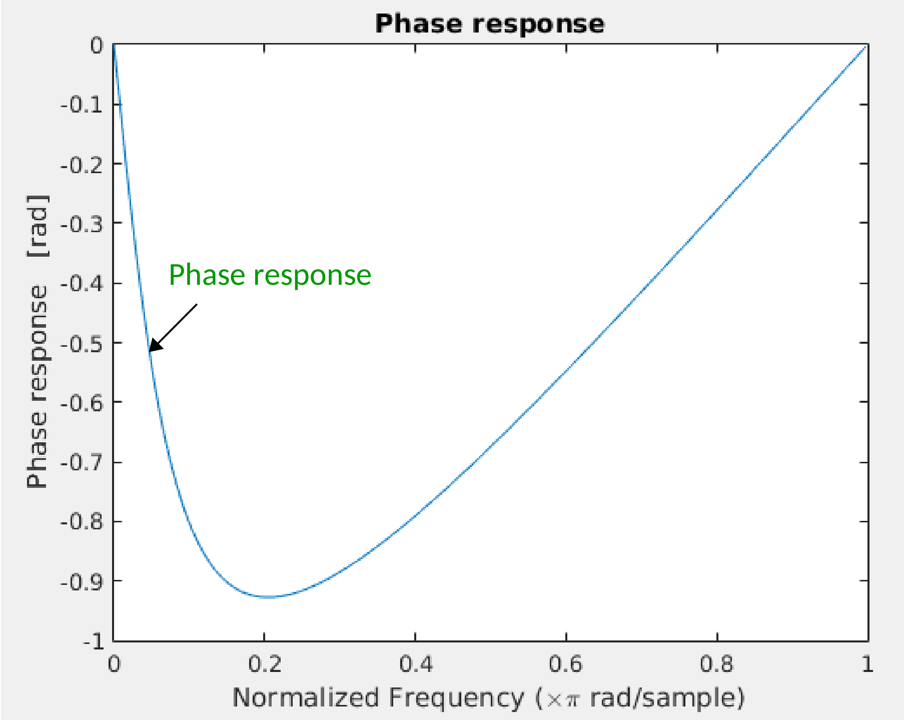
- Example, "leaky integrator", the first-order lowpass filter:
- Frequency response

$$H(e^{j\omega}) = \frac{b_0\, e^{j\omega}}{(e^{j\omega} - a)} \qquad a = 0.8$$

- Phase response

$$\theta(\omega) = \arg\{e^{j\omega}\} - \arg\{e^{j\omega} - a\}$$

$$\theta(\omega) = \omega - \arctan\frac{\sin\omega}{\cos\omega - a}$$

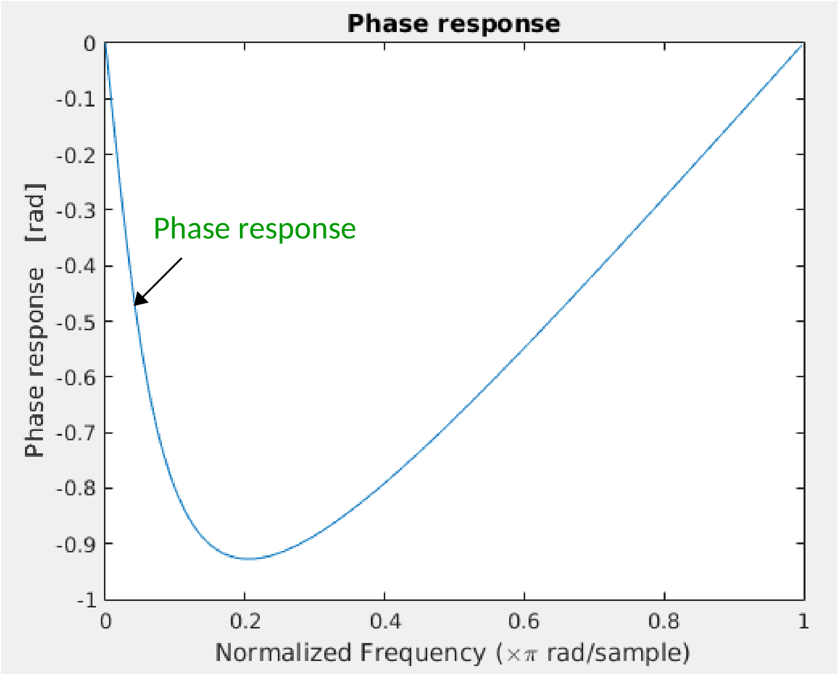


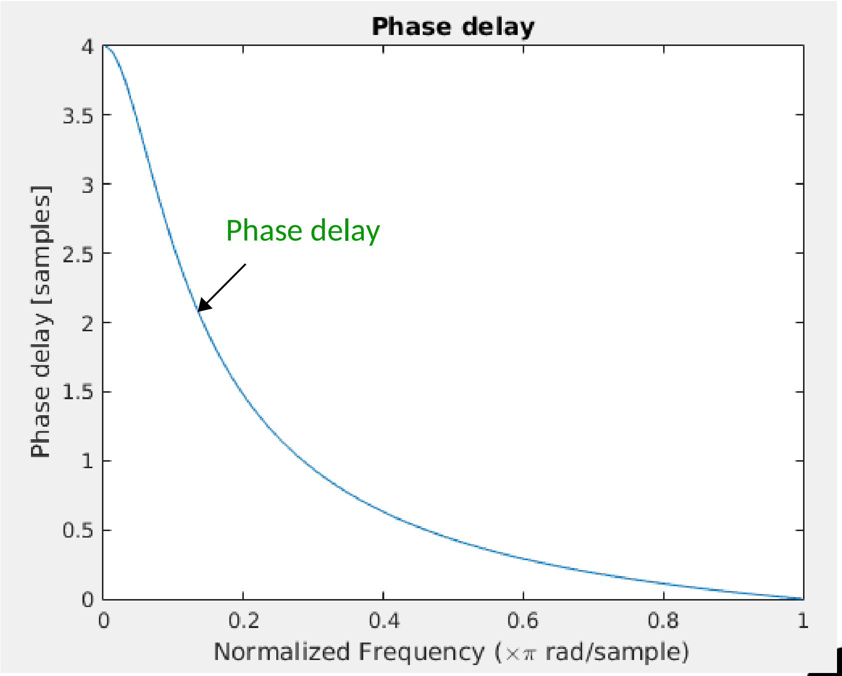
Infinite Impulse Response (IIR) systems

- Example, “leaky integrator”, the first-order lowpass filter:
- Phase response — Phase delay

$$\theta(\omega) = \omega - \arctan\frac{\sin\omega}{\cos\omega - a}$$

$$\tau_P(\omega) = \frac{-\theta(\omega)}{\omega}$$





Group delay

- Group delay is the negative first derivative of the phase response (i.e., rate of change of the phase response at a frequency point)
  - The group delay gives the time delay in seconds of narrowband frequency packets of the input signal

$$\tau_G(\omega) = \frac{-d\,\theta(\omega)}{d\,\omega}$$

  - For linear phase responses, the group delay and the phase delay are identical
  - Consider a broadband signal as a superposition of narrowband signals (frequency packets) with different center frequencies

    → Non-linearity of the phase response results in a dispersion in time of the frequency packets in the output signal

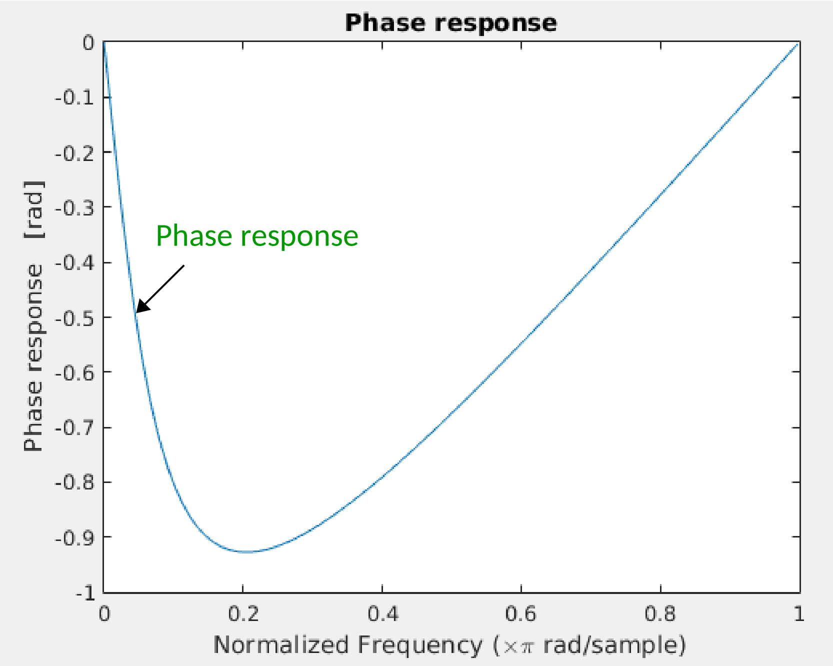
Infinite Impulse Response (IIR) systems

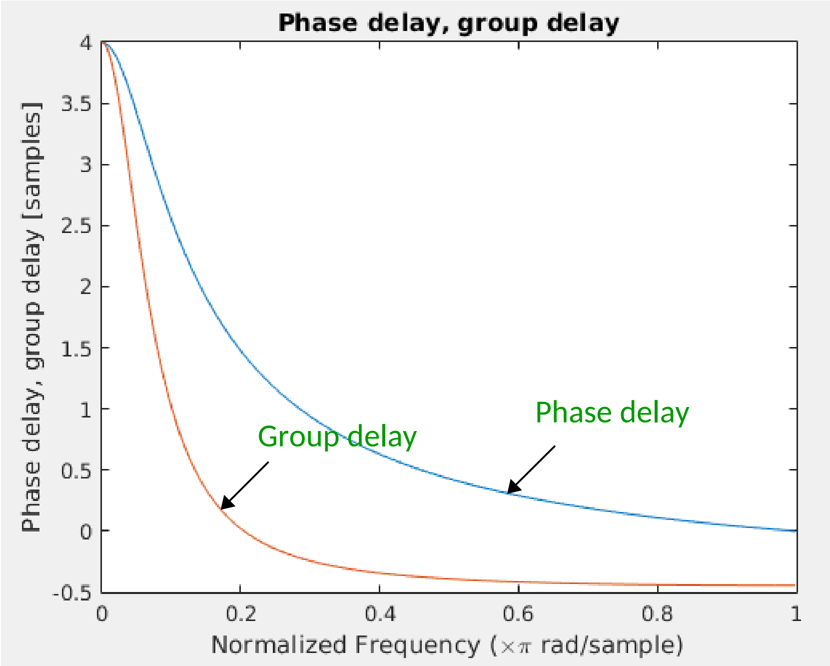
- Example, “leaky integrator”, the first-order lowpass filter:
- Phase response

$$\theta(\omega) = \omega - \arctan \frac{\sin\omega}{\cos\omega - a}$$

Phase delay and group delay

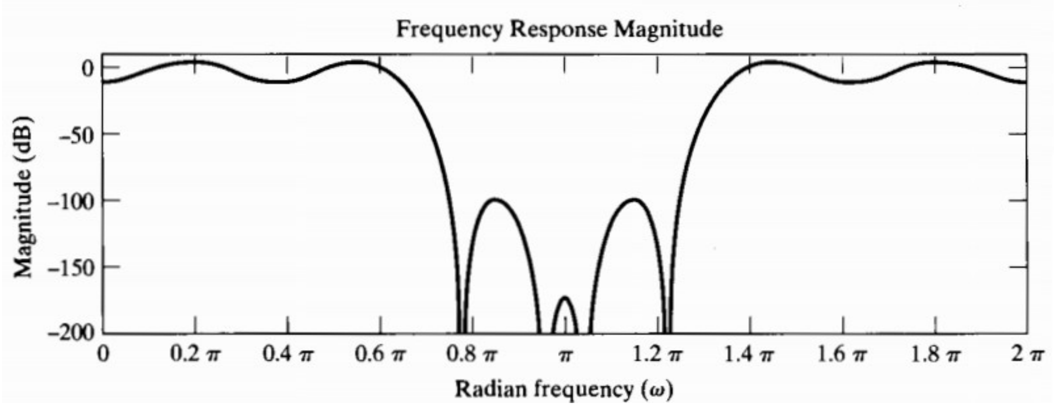
$$\tau_P(\omega) = \frac{-\theta(\omega)}{\omega} \qquad \tau_G(\omega) = \frac{-d\,\theta(\omega)}{d\,\omega}$$





Infinite Impulse Response (IIR) systems

- Illustration of effects of group delay and attenuation
- Amplitude response of IIR system



- Input signal

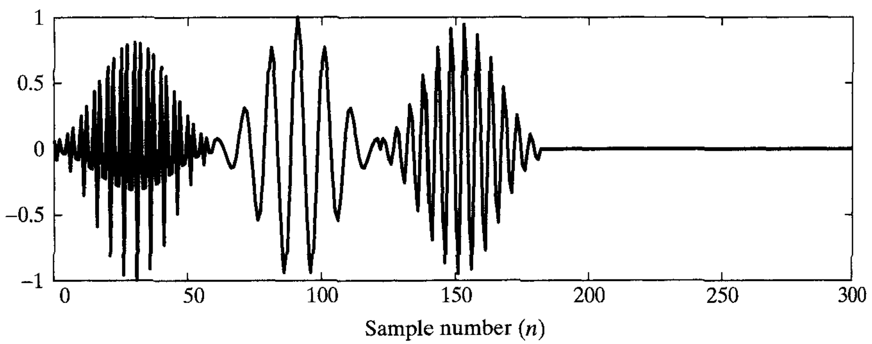
$$x(n) = x_1(n-M-1) + x_2(n-2M-2) + x_3(n)$$

$$x_1(n) = w(n)\cos(0.25\pi n)$$

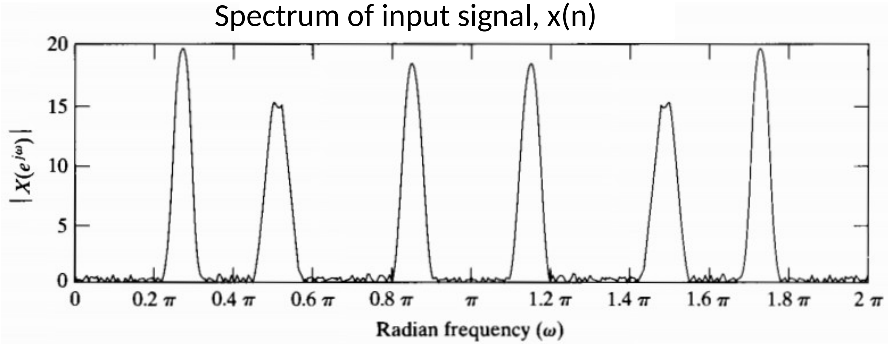
$$x_2(n) = w(n)\cos\left(0.5\pi n - \frac{\pi}{2}\right)$$

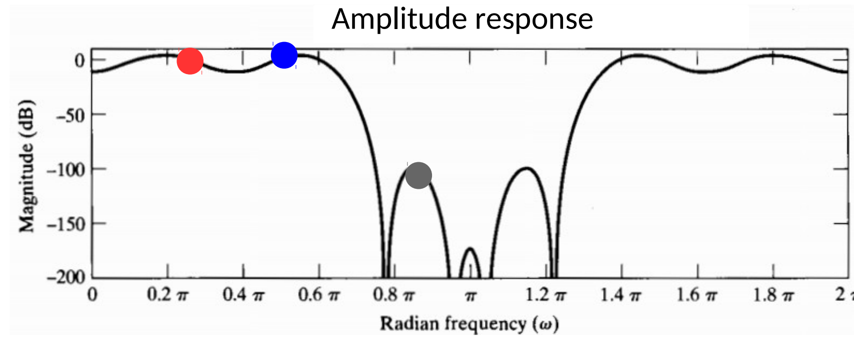
$$x_3(n) = w(n)\cos\left(0.85\pi n + \frac{\pi}{5}\right)$$

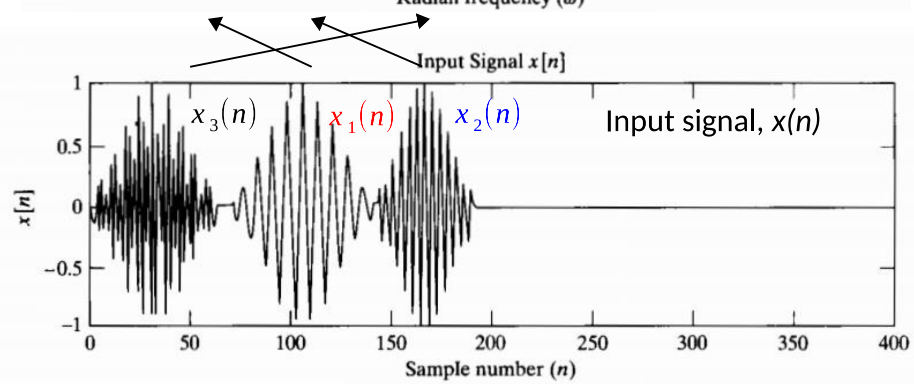
$$w(n)=\begin{cases}0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M\\ 0, & \text{otherwise}\end{cases}$$

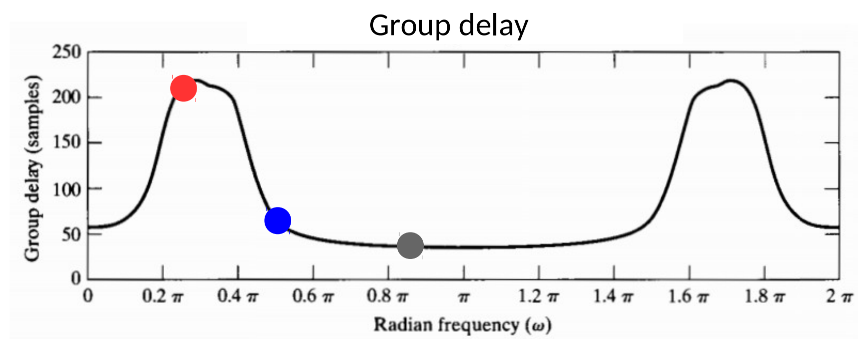


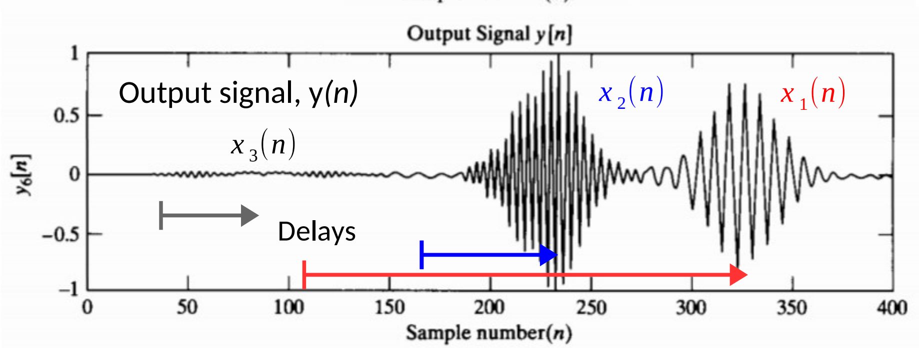
Infinite Impulse Response (IIR) systems











$x_1(n)$: $\omega_1 = 0.25\pi$, Delay of $x_1(n) \approx 215$ samples

$x_2(n)$: $\omega_2 = 0.50\pi$, Delay of $x_2(n) \approx 70$ samples

$x_3(n)$: $\omega_3 = 0.85\pi$, Delay of $x_3(n) \approx 45$ samples

Attenuation ?

[Oppenheim, Schafer]

Stability

- A causal LTI system is BIBO (Bounded Input Bounded Output) stable if and only if its poles are inside the unit circle

Bouded Input Bounded Output:

$$\exists B_x,\, B_y: \quad |x(n)| \leqslant B_x < \infty, \quad |y(n)| \leqslant B_y < \infty$$

Stability

- Example of exam task

Causal linear time-invariant system has the following transfer function:

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{1}{2} z^{-2}}$$

Find the difference equation of this system. Determine the zeros and poles of this system. Sketch the zero-pole diagram in the Z plane. Is this system stable in BIBO sense? You have to justify your answer. Sketch also the amplitude response of this filter.

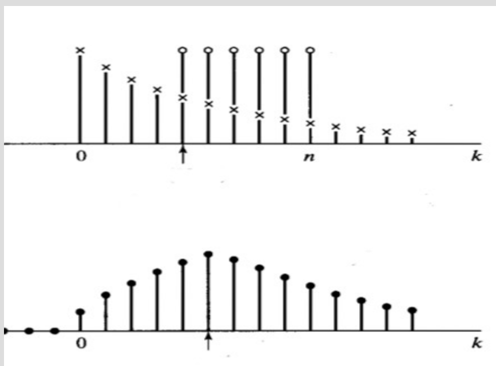
Review of LTI systems – the Big Picture

IR

$$y(n) = x(n) * h(n)$$

$x(n)*h(n)$

$y(n)$



DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

IDTFT

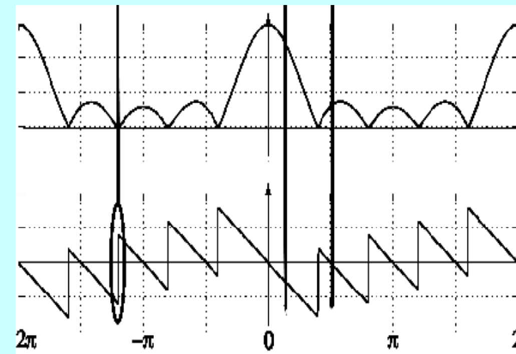
$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

DTFT

$$Y(e^{j\omega}) = X(e^{j\omega}) . H(e^{j\omega})$$

$|H(e^{j\omega})|$

$|\theta(\omega)|$



$y(n)$

$$a^n u(n) \leftrightarrow \frac{1}{1 - a z^{-1}}$$

ZT

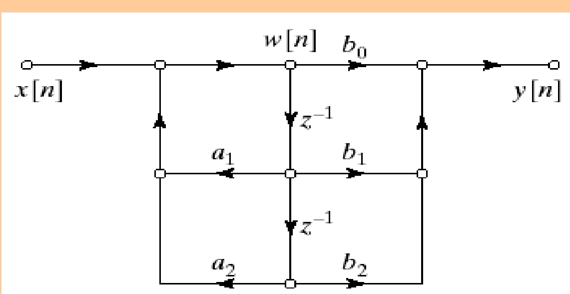
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

IZT

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$$

LCCDE

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$



ZT

$$\sum_j a_j x(n-j) \leftrightarrow \sum_j a_j z^{-j} X(z)$$

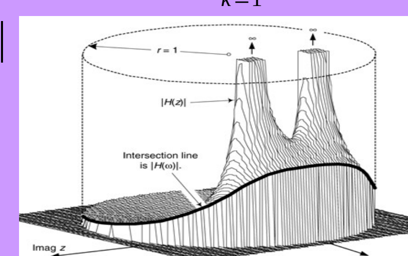
$$\sum_j b_j y(n-j) \leftrightarrow \sum_j b_j z^{-j} Y(z)$$

IZT

ZT

$$Y(z) = \frac{b_0}{a_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)} X(z)$$

$|H(z)|$



(Additional materials)

- Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) systems
- Finite Impulse Response (FIR) systems
- Geometric interpretation
- Infinite impulse response (IIR) systems
- Stability
- Phase delay, group delay, MATLAB
- Stability, MATLAB

Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) systems

- FIR filters

  - If all the a_k coefficients are zero

    * The output depends only on a finite number of values of the input. Termed also as all-zero, or moving average (MA) filters.

- IIR filters

  - If at least one of the a_k coefficients is nonzero:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

  (a) Autoregressive (AR) filters

  If all of the b_k coefficients except b_0 are zero, the output depends only on the current value of the input and a finite number of past values of the output. Termed also as all-pole, purely recursive, or autoregressive (AR) filters. The term “autoregressive” means that the output is approximately a sum of its own past values.

  (b) Autoregressive, moving-average (ARMA) filters

  Both a_k and b_k coefficients are nonzero, with $N \geq 1$ and $M > 0$.
  Also termed as pole-zero or autoregressive, moving average (ARMA) filters.

Finite Impulse Response (FIR) systems

- Example

Consider the finite impulse response, *h(n)*

$$h(n) = \begin{cases} a^n, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

$M = 8, \ |a| < 1$

- The transfer function is

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = \sum_{n=0}^{M-1} a^n z^{-n} = \frac{1 - a^M z^{-M}}{1 - a z^{-1}}$$

- Assuming that *a* is real and positive, the zeros, $z_{(k+1)}$, can be written as

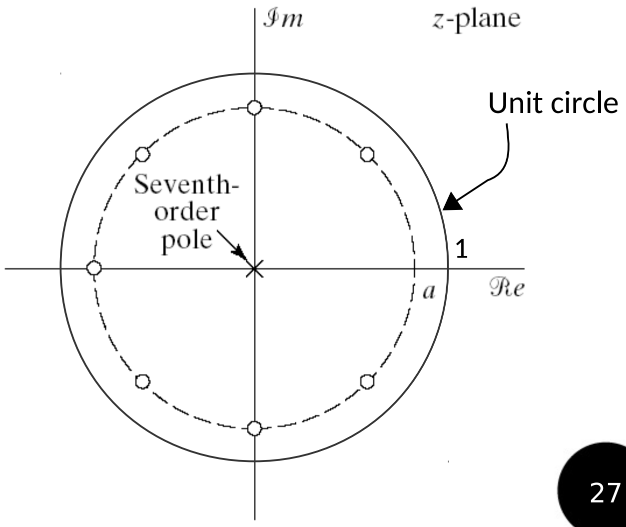
$$z_{(k+1)} = a e^{j 2\pi k / M}, \quad k = 0,1,\ldots,M-1$$

- For $k = 0$ we have a zero at $z_0 = a$

- The zero cancels the pole at $p_1 = a$

- The output, *y(n)*, can be written as

$$y(n) = \sum_{k=0}^{M-1} a^k x(n-k) \qquad y(n) = a\, y(n-1) + x(n) - a^M x(n-M)$$



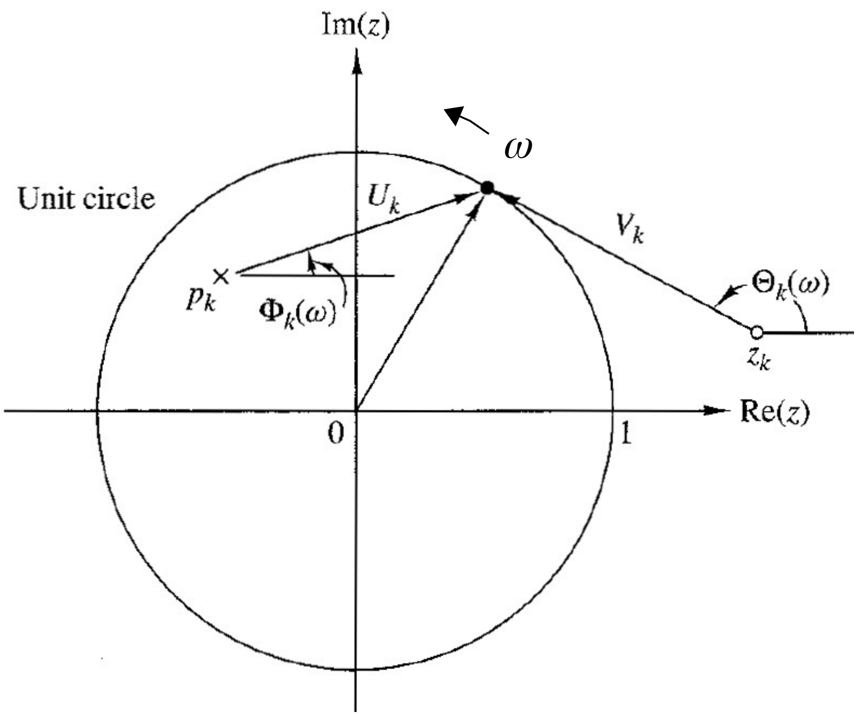
- *To plot zeros and poles, amplitude and phase response, and phase and group delay, use MATLAB. ROC?, Stability?*

Geometric interpretation

- *The amplitude of frequency response*, $|H(e^{j\omega})|$, is product of length of vectors from zeros, V_k, divided by product of lengths of vectors from poles, U_k

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} |e^{j\omega} - z_k|}{\prod_{k=1}^{N} |e^{j\omega} - p_k|} = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} V_k(e^{j\omega})}{\prod_{k=1}^{N} U_k(e^{j\omega})}$$

- *The phase of frequency response*, $\theta(\omega)$, is sum of angles Θ_k of vectors from zeros, minus sum of angles Φ_k of vectors from poles



$$\theta(\omega) = \arg\{\frac{b_0}{a_0}\} + \omega(N-M) + \sum_{k=1}^{M} \arg\{e^{j\omega} - z_k\} - \sum_{k=1}^{N} \arg\{e^{j\omega} - p_k\}$$

$$\theta(\omega) = \arg\{\frac{b_0}{a_0}\} + \omega(N-M) + \sum_{k=1}^{M} \Theta_k(\omega) - \sum_{k=1}^{N} \Phi_k(\omega)$$

[Proakis, Manolakis]

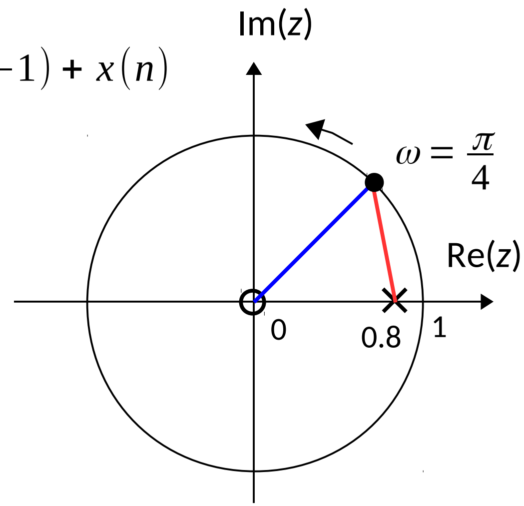
Geometric interpretation

- Example $\quad y(n) = 0.8\,y(n-1) + x(n)$

$$H(z) = \frac{1}{1-0.8\,z^{-1}} = \frac{z}{z-0.8}$$

$$H(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega}-0.8}$$

$$z_1 = 0, \qquad p_1 = 0.8$$

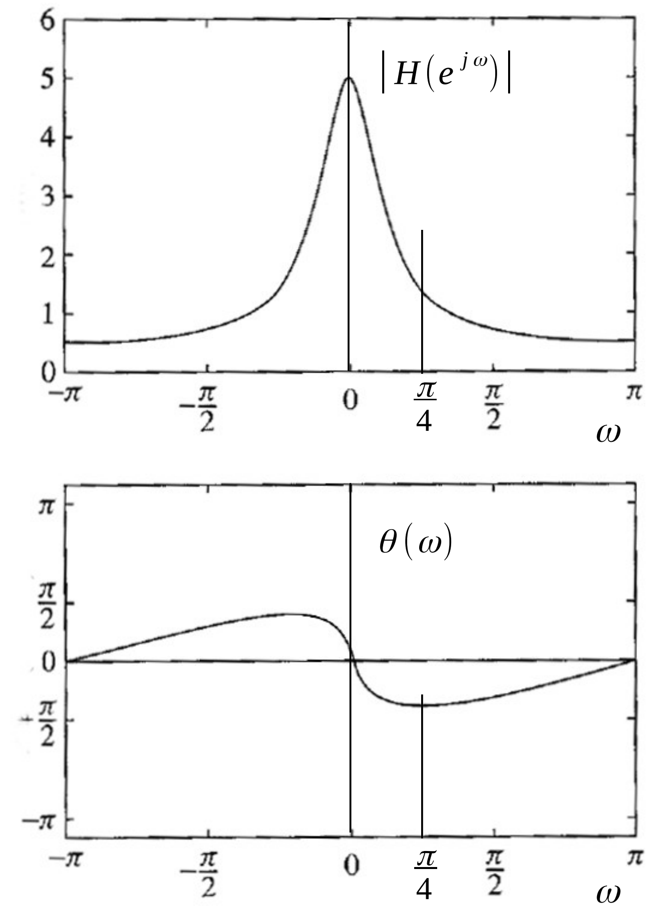


$$|H(e^{j\omega})| = \frac{V_1(e^{j\omega})}{U_1(e^{j\omega})} = \frac{|e^{j\omega}|}{|e^{j\omega}-0.8|} \qquad |H(0)| = \frac{1}{0.2} = 5$$

$$|H(e^{j\omega})| = \frac{1}{\sqrt{1.64-1.6\cos\omega}} \qquad \left|H\left(\frac{\pi}{4}\right)\right| = \frac{1}{0.71} = 1.4$$

$$\theta(\omega) = \Theta_1(\omega) - \Phi_1(\omega) \qquad \theta(0) = 0 - 0 = 0$$

$$\theta(\omega) = \omega - \arctan\frac{\sin\omega}{\cos\omega - 0.8} \qquad \theta\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - 1.7 = -0.91$$



For a demo see: https://engineering.purdue.edu/VISE/ee438/demos/

[Proakis, Manolakis]

Infinite Impulse Response (IIR) systems

- Example, 2nd order IIR system, a digital resonator
- Transfer function

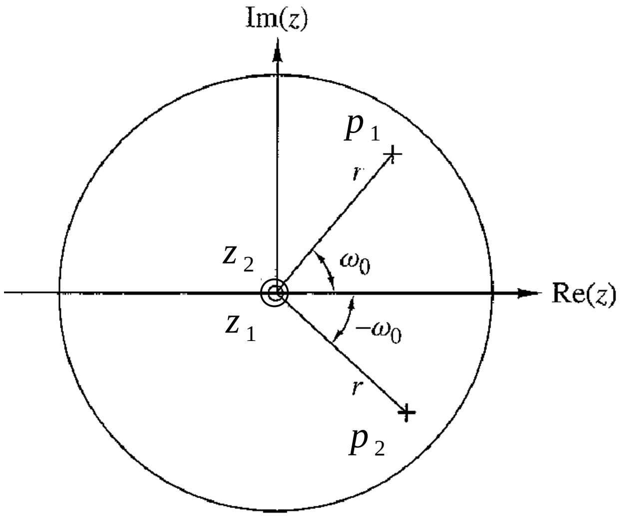
$$H(z) = \frac{b_0}{a_0}\frac{z^2}{(z-p_1)(z-p_2)}, \qquad z_{1,2}=0, \quad p_{1,2}=r\,e^{\pm j\omega_0}, \quad a_0=1$$

$$H(z) = b_0\frac{z^2}{z^2}\frac{1}{(1-p_1 z^{-1})(1-p_2 z^{-1})} = \frac{b_0}{(1-r\,e^{j\omega_0}z^{-1})(1-r\,e^{-j\omega_0}z^{-1})}$$

$$H(z) = \frac{b_0}{1-(p_1+p_2)z^{-1}+p_1 p_2\, z^{-2}}$$

$$H(z) = \frac{b_0}{1-(2r\cos\omega_0)z^{-1}+r^2 z^{-2}}$$

Real numbers



- The output, *y(n)*, can be written as

$$y(n) = (2r\cos\omega_0)\,y(n-1) - r^2 y(n-2) + b_0 x(n)$$

[Proakis, Manolakis]

Infinite Impulse Response (IIR) systems

- Example, 2nd order IIR system, a digital resonator

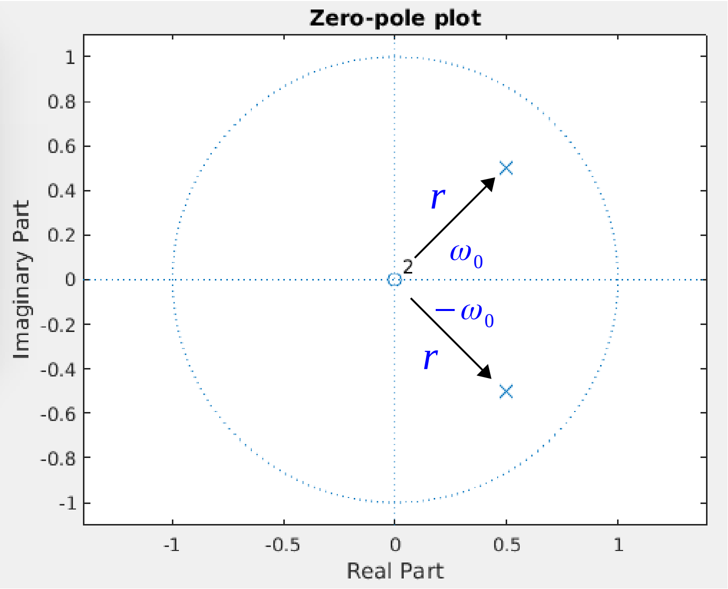
$$H(z) = \frac{b_0}{a_0}\frac{z^2}{(z-p_1)(z-p_2)}, \quad z_{1,2}=0, \quad p_{1,2}=r\,e^{\pm j\omega_0}, \quad a_0=1, \quad r=\frac{\sqrt{2}}{2}, \quad \omega_0=\frac{\pi}{4}$$

$$H(z) = \frac{b_0}{1-(2r\cos\omega_0)\,z^{-1}+r^2 z^{-2}} = \frac{b_0}{1-z^{-1}+0.5\,z^{-2}} \quad \rightarrow \quad y(n) = y(n-1) - 0.5\,y(n-2) + b_0 x(n)$$

- Transfer function

$$H(z) = \frac{b_0\; z\; z}{(z-r\,e^{j\omega_0})(z-r\,e^{-j\omega_0})}$$

$$H(z) = \frac{b_0\; z\; z}{\left(z-\frac{\sqrt{2}}{2}e^{j\frac{\pi}{4}}\right)\left(z-\frac{\sqrt{2}}{2}e^{-j\frac{\pi}{4}}\right)}$$



Infinite Impulse Response (IIR) systems

- Example, 2nd order IIR system, a digital resonator

$$H(z) = \frac{b_0}{a_0}\frac{z^2}{(z-p_1)(z-p_2)},\quad z_{1,2}=0,\quad p_{1,2}=r\,e^{\pm j\omega_0},\quad a_0=1,\quad r=\frac{\sqrt{2}}{2},\quad \omega_0=\frac{\pi}{4}$$

$$H(z) = \frac{b_0}{1-(2r\cos\omega_0)\,z^{-1}+r^2z^{-2}} = \frac{b_0}{1-z^{-1}+0.5\,z^{-2}} \rightarrow y(n)=y(n-1)-0.5\,y(n-2)+b_0x(n)$$

- Frequency response

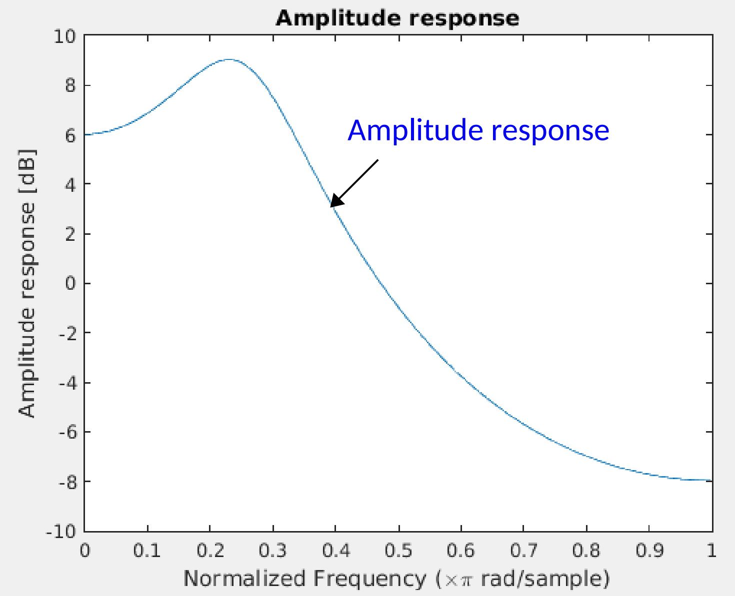
$$H(e^{j\omega}) = \frac{b_0\,e^{j\omega}\,e^{j\omega}}{(e^{j\omega}-r\,e^{j\omega_0})(e^{j\omega}-r\,e^{-j\omega_0})}$$

- Amplitude response $\quad (\,b_0 = 1\,)$

$$|H(e^{j\omega})| = \frac{b_0\,|e^{j\omega}||e^{j\omega}|}{|e^{j\omega}-r\,e^{j\omega_0}||e^{j\omega}-r\,e^{-j\omega_0}|} = \frac{b_0}{U_1(e^{j\omega})\,U_2(e^{j\omega})}$$

$$U_1(e^{j\omega}) = \sqrt{1+r^2-2r\cos(\omega_0+\omega)}$$

$$U_2(e^{j\omega}) = \sqrt{1+r^2-2r\cos(\omega_0-\omega)}$$



Infinite Impulse Response (IIR) systems

$$r = \frac{\sqrt{2}}{2}, \quad \omega_0 = \frac{\pi}{4}$$

- Example, 2nd order IIR system, a digital resonator
- Frequency response

$$H(e^{j\omega}) = \frac{b_0 \, e^{j\omega} e^{j\omega}}{(e^{j\omega} - r e^{j\omega_0})(e^{j\omega} - r e^{-j\omega_0})}$$

- Amplitude response

$$|H(e^{j\omega})| = \frac{b_0 \, |e^{j\omega}| |e^{j\omega}|}{|e^{j\omega} - r e^{j\omega_0}| |e^{j\omega} - r e^{-j\omega_0}|}$$

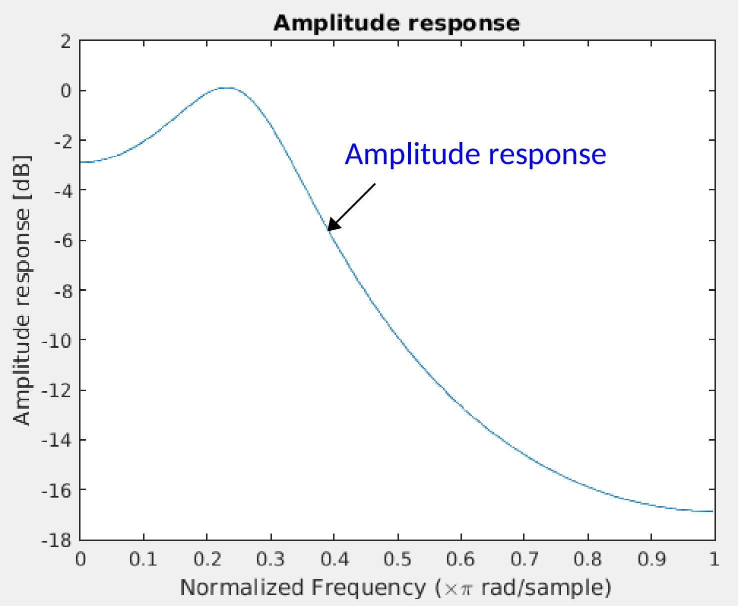
- Normalization factor, b_0

$$|H(e^{j\omega_0})| = \frac{b_0 \, |e^{j\omega_0}| |e^{j\omega_0}|}{|e^{j\omega_0} - r e^{j\omega_0}| |e^{j\omega_0} - r e^{-j\omega_0}|} = 1$$

$$|H(e^{j\omega_0})| = \frac{b_0}{|1 - r| |1 - r e^{-j2\omega_0}|} = 1$$

$$|H(e^{j\omega_0})| = \frac{b_0}{(1 - r)\sqrt{1 + r^2 - 2r\cos(2\omega_0)}} = 1$$

$$b_0 = \left(1 - \frac{\sqrt{2}}{2}\right)\sqrt{1 + \frac{1}{2} - 2\frac{\sqrt{2}}{2}\cos\left(2\frac{\pi}{4}\right)} = 0.3587$$



Infinite Impulse Response (IIR) systems

- Example, 2nd order IIR system, a digital resonator

$$r = \frac{\sqrt{2}}{2}, \quad \omega_0 = \frac{\pi}{4}$$

- Frequency response

$$H(e^{j\omega}) = \frac{b_0\, e^{j\omega}\, e^{j\omega}}{(e^{j\omega} - r\,e^{j\omega_0})(e^{j\omega} - r\,e^{-j\omega_0})}$$

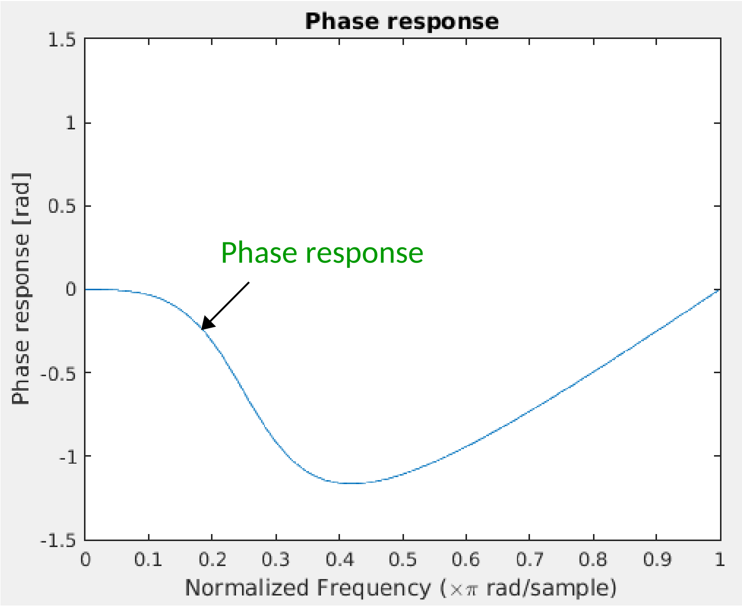
- Phase response

$$\theta(\omega) = \arg\{e^{j\omega}\} + \arg\{e^{j\omega}\} - \arg\{e^{j\omega} - r\,e^{j\omega_0}\} - \arg\{e^{j\omega} - r\,e^{-j\omega_0}\}$$

$$\theta(\omega) = \omega + \omega - \Phi_1(\omega) - \Phi_2(\omega)$$

$$\Phi_1(\omega) = \arctan\left[\frac{\sin\omega - r\sin\omega_0}{\cos\omega - r\cos\omega_0}\right]$$

$$\Phi_2(\omega) = \arctan\left[\frac{\sin\omega + r\sin\omega_0}{\cos\omega - r\cos\omega_0}\right]$$



Infinite Impulse Response (IIR) systems

- Example, 2nd order IIR system, a digital resonator

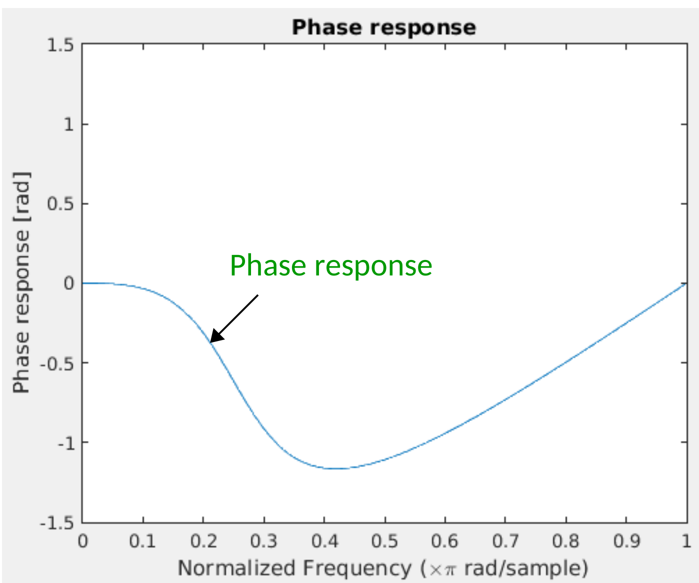
$$r = \frac{\sqrt{2}}{2}, \quad \omega_0 = \frac{\pi}{4}$$

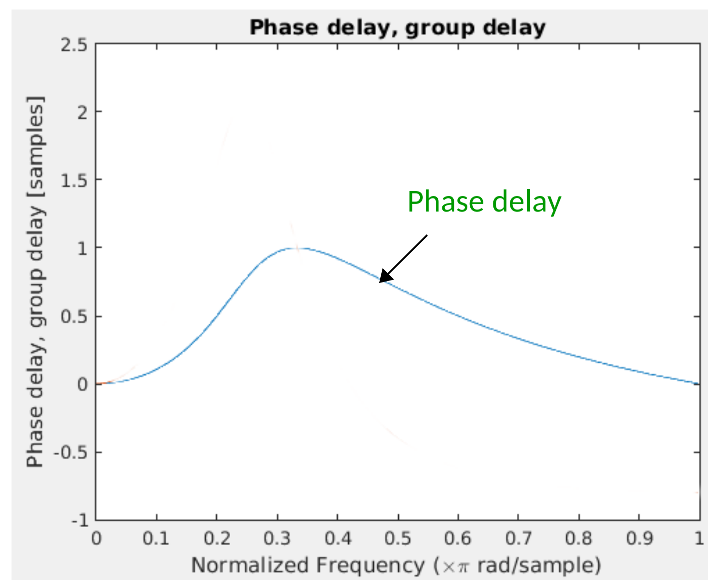
- Phase response

$$\theta(\omega) = \omega + \omega - \arctan\left[\frac{\sin\omega - r\sin\omega_0}{\cos\omega - r\cos\omega_0}\right] - \arctan\left[\frac{\sin\omega + r\sin\omega_0}{\cos\omega - r\cos\omega_0}\right]$$

Phase delay

$$\tau_P(\omega) = \frac{-\theta(\omega)}{\omega}$$





Infinite Impulse Response (IIR) systems

- Example, 2nd order IIR system, a digital resonator

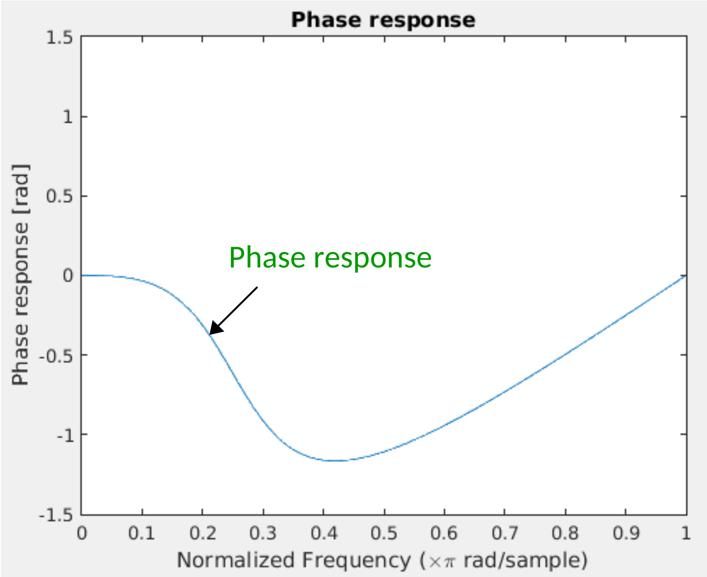
$$r = \frac{\sqrt{2}}{2}, \quad \omega_0 = \frac{\pi}{4}$$

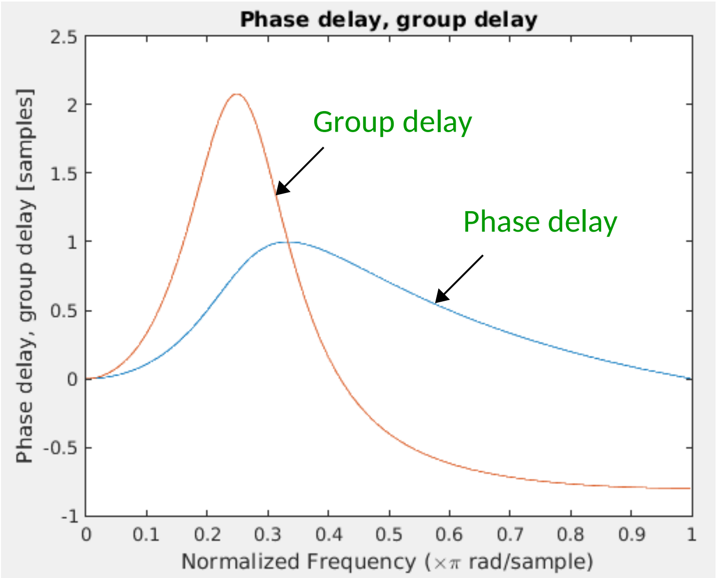
- Phase response

$$\theta(\omega) = \omega + \omega - \arctan\left[\frac{\sin\omega - r\sin\omega_0}{\cos\omega - r\cos\omega_0}\right] - \arctan\left[\frac{\sin\omega + r\sin\omega_0}{\cos\omega - r\cos\omega_0}\right]$$

Phase delay and group delay

$$\tau_P(\omega) = \frac{-\theta(\omega)}{\omega} \qquad \tau_G(\omega) = \frac{-d\theta(\omega)}{d\omega}$$





Stability

- An LTI system is said to be Bounded Input Bounded Output (BIBO) stable if and only if every bounded input produces a bounded output

$$\exists B_x, B_y: \quad |x(n)| \leqslant B_x < \infty, \quad |y(n)| \leqslant B_y < \infty$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$|y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right|$$

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)|$$

$$|y(n)| \leq B_x \sum_{k=-\infty}^{\infty} |h(k)| \leq B_y$$

- An LTI system is stable ($B_y < \infty$) if and only if $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

  → (Absolute summability of *h(k)* implies the existence of DTFT)

- The system can give meaningful output only if it is stable

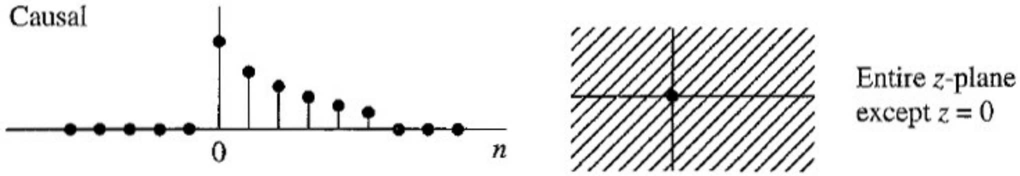
Stability

- A transfer function does not uniquely specify a system (need to know ROC)
- Properties of system suggests properties of ROC

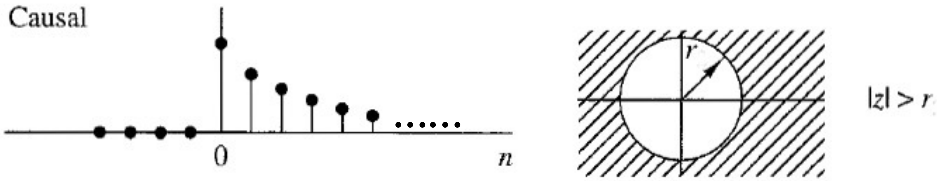
  - For causal LTI systems (right sided), the impulse response satisfies the following condition

$$h(n) = 0, \quad \text{for} \quad n < 0$$

  - ROC for finite-duration impulse responses (*z* plane, except *z* = 0)



  - ROC for infinite-duration impulse responses (|*z*| > *r*)



- An LTI system is causal if and only if the ROC of the transfer function is the exterior of circle of radius $r < \infty$, including the point $z = \infty$ (1.)

Stability

- Stability of an LTI system expressed in terms of the characteristics of the transfer function, *H(z)*
- A necessary and sufficient condition for an LTI system to be BIBO stable is

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

- This condition implies that *H(z)* must contain the unit circle within its ROC

Since $$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

follows $$|H(z)| = \left|\sum_{n=-\infty}^{\infty} h(n)\, z^{-n}\right| \le \sum_{n=-\infty}^{\infty} |h(n)\, z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)|\,|z^{-n}|$$

Evaluating on the unit circle ($|z| = 1$) $\rightarrow |H(z)| \le \sum_{n=-\infty}^{\infty} |h(n)| \rightarrow |H(e^{j\omega})|$ exists

→ The unit circle is contained in the ROC of *H(z)* → (DTFT exists)

- An LTI system is BIBO stable if and only if the ROC of the transfer function, *H(z)*, includes the unit circle (2.)

Stability

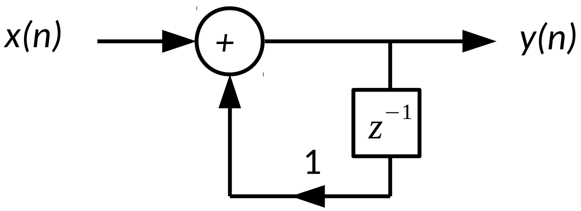
- A causal (1.) LTI system is BIBO (Bounded Input Bounded Output) stable (2.) if and only if its poles are inside the unit circle, $|p_k| < 1$, for $k = 1,2, \dots, N$
  - The region of convergence (ROC) cannot contain any pole since *H(z)* is infinitely large at any pole
  - (1.) An LTI system is causal if and only if the ROC of the transfer function is the exterior of circle of radius $r < \infty$, including the point $z = \infty$
  - (2.) An LTI system is BIBO stable if and only if the ROC of the transfer function, *H(z)*, includes the unit circle

Bouded Input Bounded Output:

$$\exists B_x, B_y: \quad |x(n)| \leqslant B_x < \infty, \quad |y(n)| \leqslant B_y < \infty$$

Stability

- An LTI system with poles on the unit circle is not stable (marginally stable)
- It produces an unbounded response when excited by an input signal that also has a pole at the same position on the unit circle
- Example, determine the step response of the following causal system



$$y(n) = y(n-1) + x(n)$$

The transfer function $H(z) = \frac{1}{1-z^{-1}}$ contains a pole at $z = 1$ (unit circle)

Input signal, $x(n) = u(n)$ unit step signal

The Z transform of *x(n)* $X(z) = \frac{1}{1-z^{-1}}$ also contains a pole at $z = 1$

Since $Y(z) = H(z)\,X(z) = \frac{1}{(1-z^{-1})^2}$ a double pole at $z = 1$

The inverse Z transform $y(n) = (n+1)\,u(n)$ which is a ramp sequence

Phase delay, group delay, MATLAB

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = K \frac{(z-z_1)(z-z_2)\ \dots\ (z-z_M)}{(z-p_1)(z-p_2)\ \dots\ (z-p_N)}$$

- MATLAB

```
>> [b, a] = zp2tf(Z, P, 1);              % Convert Zero-pole-gain parameters to Transfer function
>>
>> [h, w] = freqz(b, a, 512);            % h - Frequency response, w – corresponding frequencies
>> [phi, w] = phasez(b, a, 512);         % phi - Phase response, w – corresponding frequencies
>>
>> [phd, w] = phasedelay(b, a, 512);     % phd - Phase delay, w – corresponding frequencies
>> [gd, w] = grpdelay(b, a, 512);        % gd - Group delay, w – corresponding frequencies
>> plot(w/pi, phi, w/pi, phd, w/pi, gd); % Plot Phase response, Phase delay, Group delay
>> ...
```

Stability, MATLAB

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = K \frac{(z-z_1)(z-z_2)\ \dots\ (z-z_M)}{(z-p_1)(z-p_2)\ \dots\ (z-p_N)}$$

- MATLAB

```
>>
>> [b, a] = zp2tf(Z, P, 1);        % Convert Zero-pole-gain parameters to Transfer function
>>
>> flag = isstable(b, a);          % If the poles are inside the unit circle, flag = true
>>
```