Transform-domain analysis of discrete-time signals and systems, I

- Introduction
- Fourier analysis of signals using DFT
- Estimating frequency spectrum
- Frequency domain analysis of signals
- Windowing
- Frequency domain analysis of signals – summary
- Frequency domain analysis of signals
- Joint time-frequency analysis
- Short Term Fourier Transform (STFT) –> spectrogram
- (Additional materials)

Introduction

- The DFT is useful tool for frequency analysis of signals
- Only signals of finite length could be analyzed
- In many spectrum analysis applications the signals inherently have finite length
- Inconsistency between the finite-length requirements of the DFT and the reality of indefinitely long signals can be accommodated through the following concepts
  - **Higher number of samples** to analyze (improving frequency domain **resolution**)
  - **Zero padding** (improving frequency domain **precision**, better representation of details)
  - **Windowing** (reducing leakage)
  - Time dependent Fourier transform (Short Term Fourier Transform → **spectrogram**)
  - (Averaging (improving the Signal-to-Noise Ratio (SNR))

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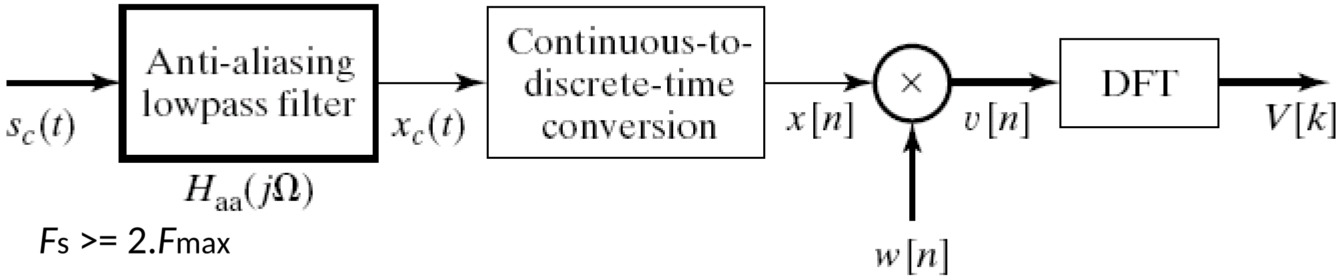
Fourier analysis of signals using DFT

- Steps in analyzing frequency content of a continuous-time signal with DFT



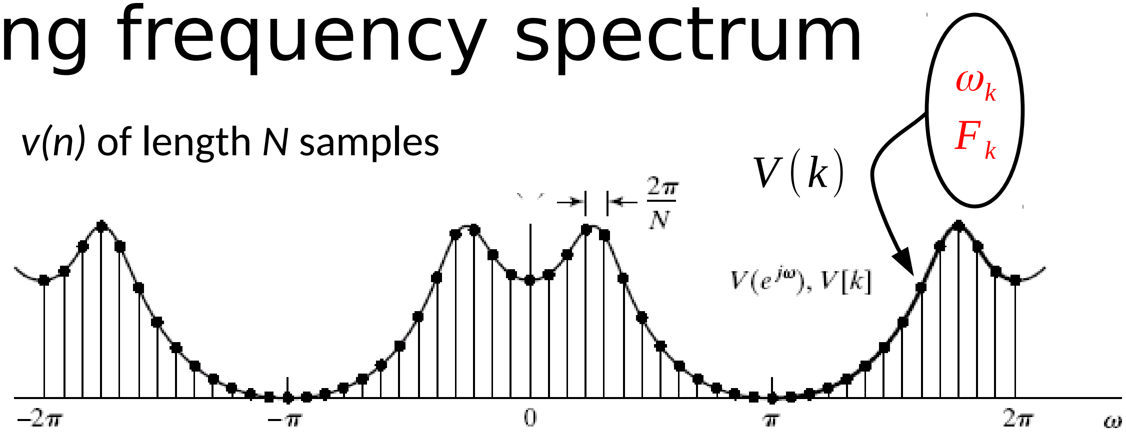
$F_s >= 2.F_{max}$

- Anti-aliasing lowpass filter
- Sample and quantize
- Discrete-time signal, *x(n)*, of infinite duration
- Window to limit the duration of the signal *x(n)*, *v(n)* = *x(n).w(n)*, where *w(n)* is the window of length *L* to limit the duration of the signal
- The signal *v(n)* could be **zero padded** to the length *N*, *i.e.*, *L* → *N*
- Take DFT of the resulting signal *v(n)* to get its spectrum *V(k)*

[Oppenheim, Schafer]

Estimating frequency spectrum

- DFT spectrum, *V(k)*, of signal *v(n)* of length *N* samples
  (*N* = *L*, <u>no zero padding</u>)



- The *k*-th component in the frequency domain is a frequency **bucket** or bin
  → Frequency *F* (or ω) of *k*-th **bucket** (each bucket corresponds to a frequency, not a range of frequencies):

$$\omega_k = k\frac{2\pi}{N}, \quad k=0,1,\ldots,N-1$$

F_S is sampling frequency

$$F_k = k\frac{F_S}{N}, \quad k=0,1,\ldots,N-1, \quad F_S \text{ relates to: } \omega = 2\pi, \; f=1 \qquad k=\frac{N}{2} \rightarrow \frac{F_S}{2}, \; \omega=\pi, \; f=\frac{1}{2}$$

- Step between two consecutive buckets (quantization in frequency)
  → Frequency resolution, ΔF_R, $\Delta\omega_R$, is proportional to the number of samples, ***N* = *L*** **!**

$$\Delta F_R = \frac{F_S}{N} \qquad \Delta\omega_R = \frac{2\pi}{N}$$

[Oppenheim, Schafer]

Estimating frequency spectrum

- Objectives
  - To improve frequency domain **resolution**
  - To improve frequency domain **precision** (zero padding, $L \rightarrow N$, *i.e.*, better representation of details)
  - To reduce leakage
  - To track changes of spectrum over time
  - (To improve Signal-to-Noise Ratio (SNR))

Frequency domain analysis of signals

- Sinusoid signal, the effect of windowing (rectangular window, length L)

$$x(n) = \cos(\omega_0 n) = \frac{1}{2} e^{-j\omega_0 n} + \frac{1}{2} e^{j\omega_0 n}$$

$$v(n) = x(n) w(n)$$

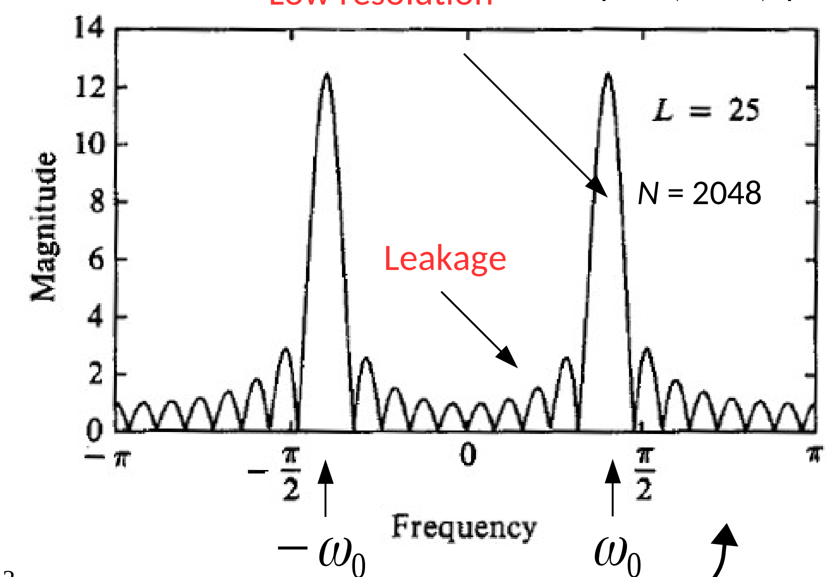
$$V(e^{j\omega}) = \text{DTFT}\left[w(n)\left(\frac{1}{2} e^{-j\omega_0 n} + \frac{1}{2} e^{j\omega_0 n}\right)\right]$$

Since $\quad e^{j\omega_0 n} x(n) \leftrightarrow X(e^{j(\omega-\omega_0)})$

$$V(e^{j\omega}) = \frac{1}{2} W(e^{j(\omega+\omega_0)}) + \frac{1}{2} W(e^{j(\omega-\omega_0)})$$

$$W(e^{j\omega}) = \text{DTFT}[w(n)] = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

$|V(e^{j\omega})|$



To approximate DTFT, zero padding, $N-L$ zeros

[Proakis, Manolakis]

Frequency domain analysis of signals

- Sinusoid signal, the effect of windowing (rectangular window, length *L*)

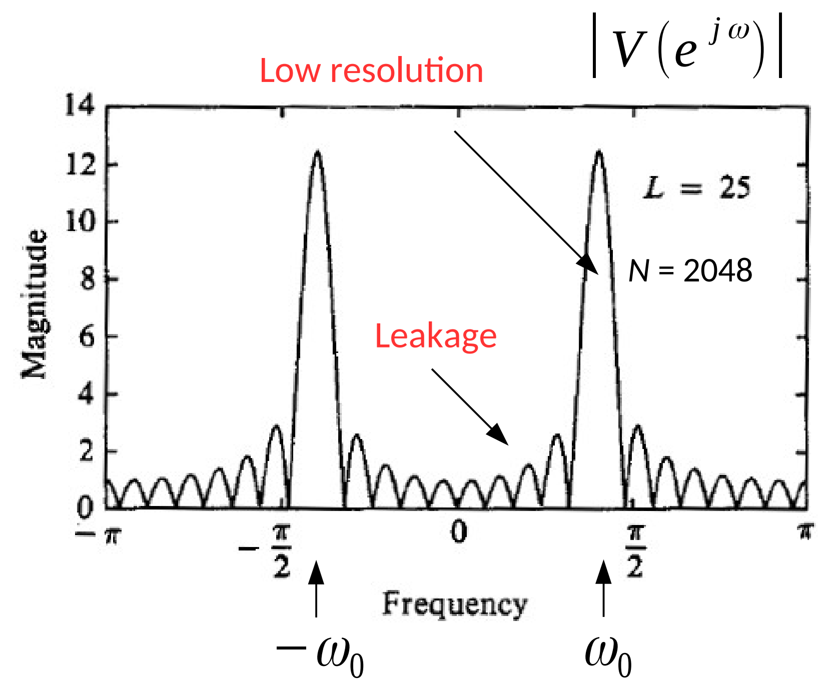
Frequency resolution: $\Delta\omega_R = \frac{2\pi}{L}$

$$\Delta\omega_R = \frac{2\pi}{L} = 0.2513\left[\frac{rad}{smp}\right]$$

Zero padding, $N-L$ zeros

Frequency precision: $\Delta\omega_P = \frac{2\pi}{N}$

$$\Delta\omega_P = \frac{2\pi}{N} = 0.0031\left[\frac{rad}{smp}\right]$$



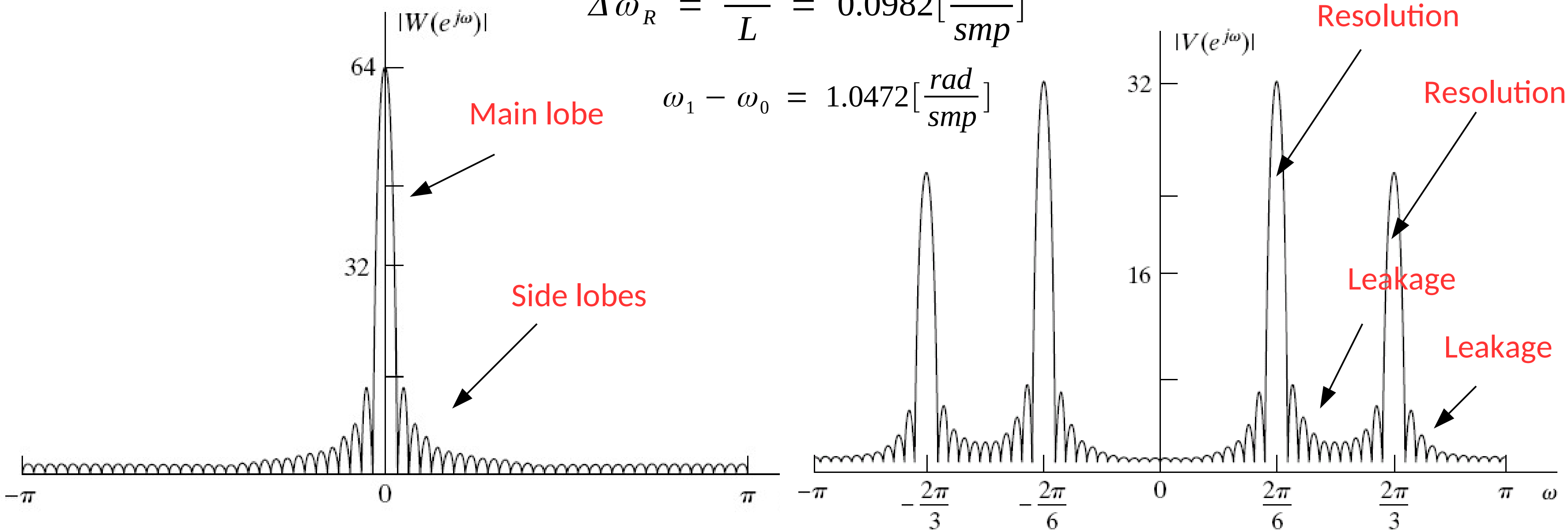
! Distinguish between *frequency resolution* (L) and *frequency precision* (N) !

Frequency domain analysis of signals

- Two sinusoids, $\omega_0 = 2\pi/6, \ \omega_1 = 2\pi/3,$ amplitudes: $A_0 = 1, \ A_1 = 0.75$
  - Rectangular window, *w(n)*, of length $L = 64$ samples
  - $Fs = 10$ k smp/sec
  - Amplitude spectra (DTFTs) of the window, *w(n)*, and of the signal, *v(n)*

$$\Delta\omega_R = \frac{2\pi}{L} = 0.0982\left[\frac{rad}{smp}\right]$$

$$\omega_1 - \omega_0 = 1.0472\left[\frac{rad}{smp}\right]$$



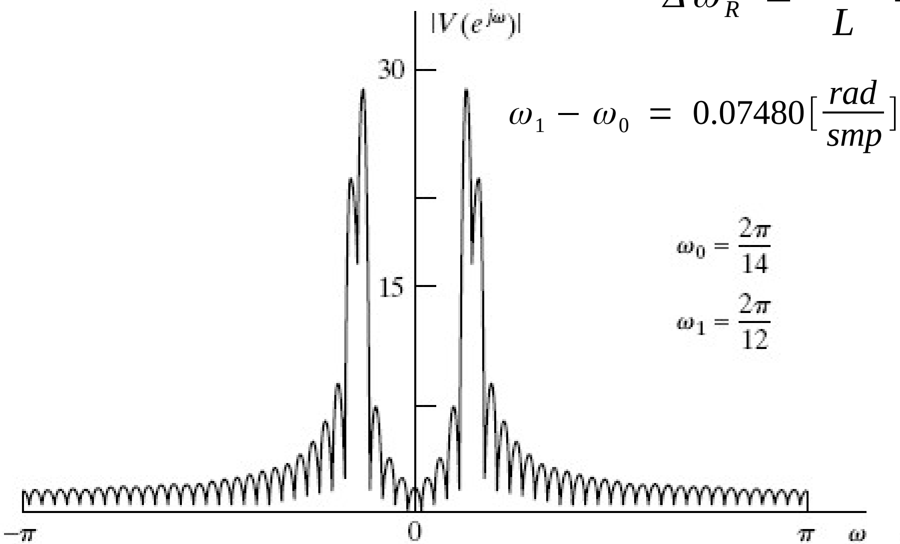
- *Assume N = 2048, Frequency precision = ? Frequency resolution, ΔF_R, = ?*

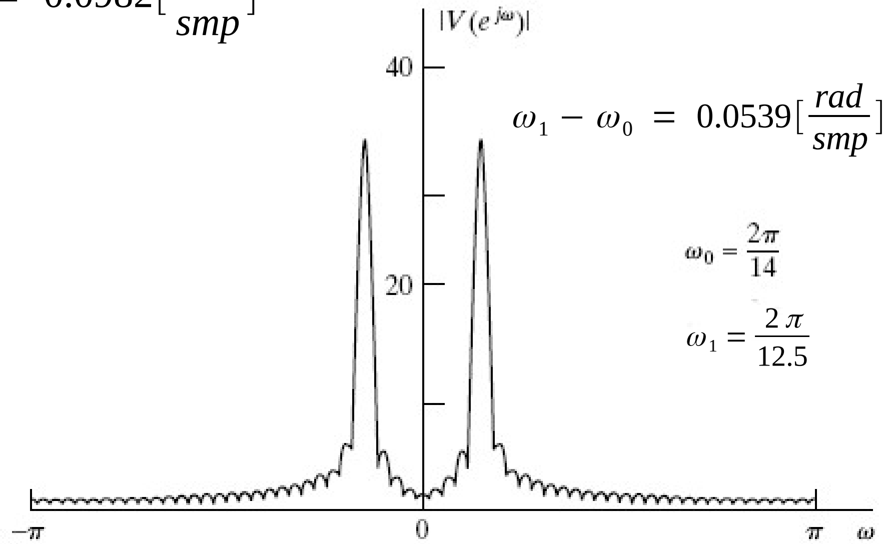
[Oppenheim, Schafer]

Frequency domain analysis of signals

- Left, two frequencies are very close, they affect each other amplitude
- Right, two frequencies are too close to be resolved (low resolution)

$$\Delta \omega_R = \frac{2\pi}{L} = 0.0982\left[\frac{rad}{smp}\right]$$





- Desired: high resolution of frequency spectrum, high frequency precision, and low leakage (low side lobes of spectrum)

Frequency domain analysis of signals

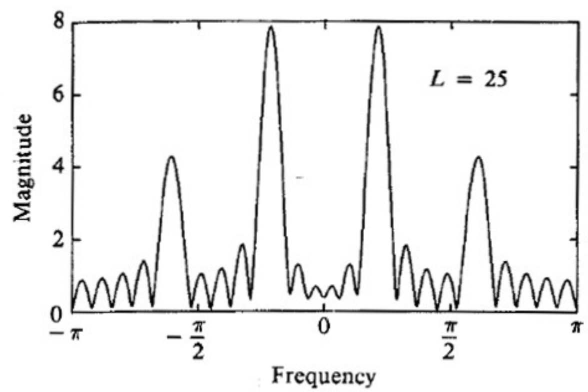
- Frequency spectrum resolution, **how to improve it?**
  - Using longer length, *L*, of weighting window, *w(n)*; example

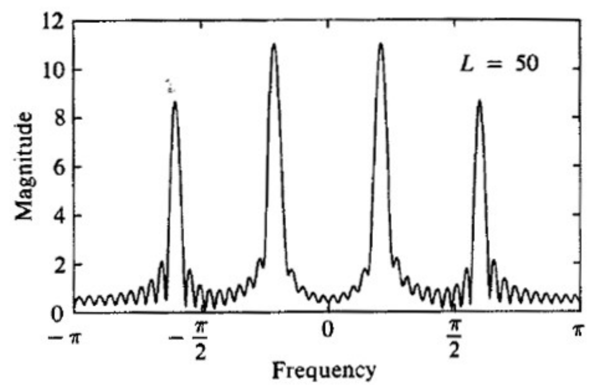
Three frequencies

$$\omega_0 = 0.2\,\pi$$

$$\omega_1 = 0.22\,\pi$$

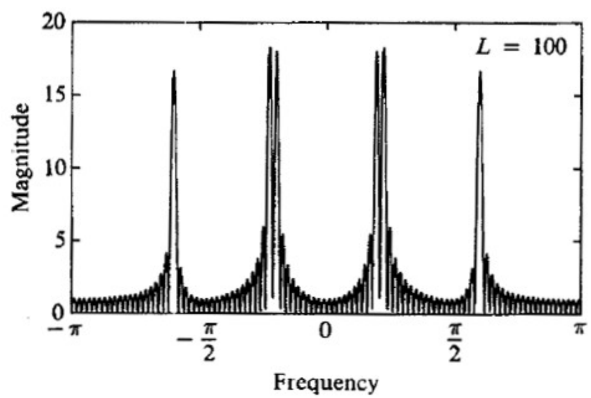
$$\omega_2 = 0.6\,\pi$$





To distinguish between two frequencies (rectangular window)

$$|\omega_i - \omega_j| \geq \Delta\omega_R = \frac{2\pi}{L}$$



[Proakis, Manolakis]

Frequency domain analysis of signals

- **Frequency precision, how to improve it?**
  - Zero padding, larger number of samples, *N*, to compute the DFT, better interpolation of spectrum only

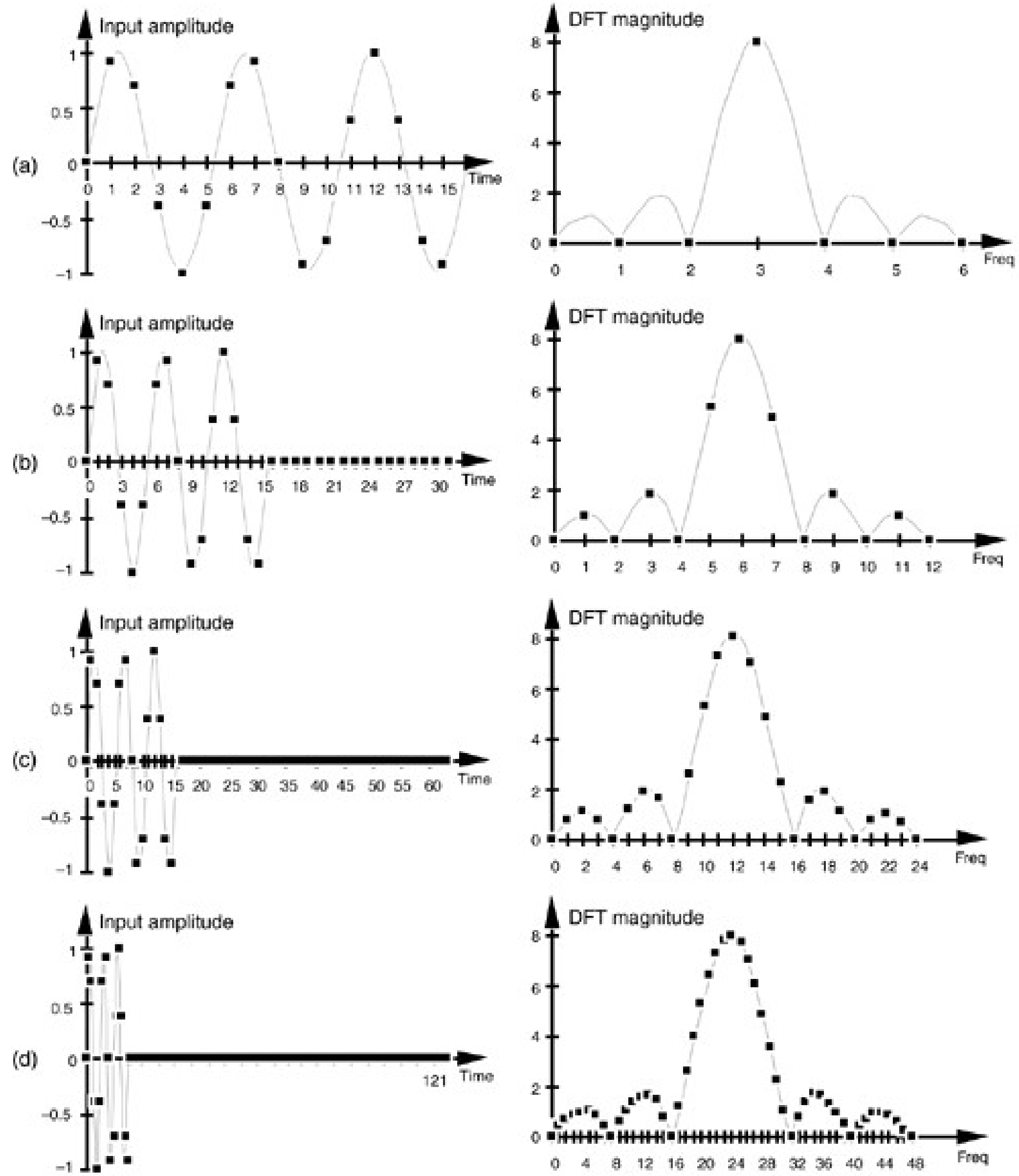
    (a) 16 samples

    (b) 32 samples (16 zeros)

    (c) 64 samples (48 zeros)

    (d) 128 samples (112 zeros)
- Task

  Calculate frequency precision in each case and frequency resolution in terms of radians



Frequency domain analysis of signals

- Example of exam task
- Discrete time signal $x(n)$ was obtained using sampling of analog signal $x_a(t)$ of length T seconds using sampling frequency $F_s = 1 / T$. How does frequency resolution ΔF_R (in Hz) of the Discrete Fourier Transform (DFT) depend on sampling frequency?

  (Assume that the length of the DFT fits the length of the signal $x(n)$.)

  A: Higher sampling frequency, F_s, increases frequency resolution.

  B: Higher sampling frequency, F_s, decreases frequency resolution.

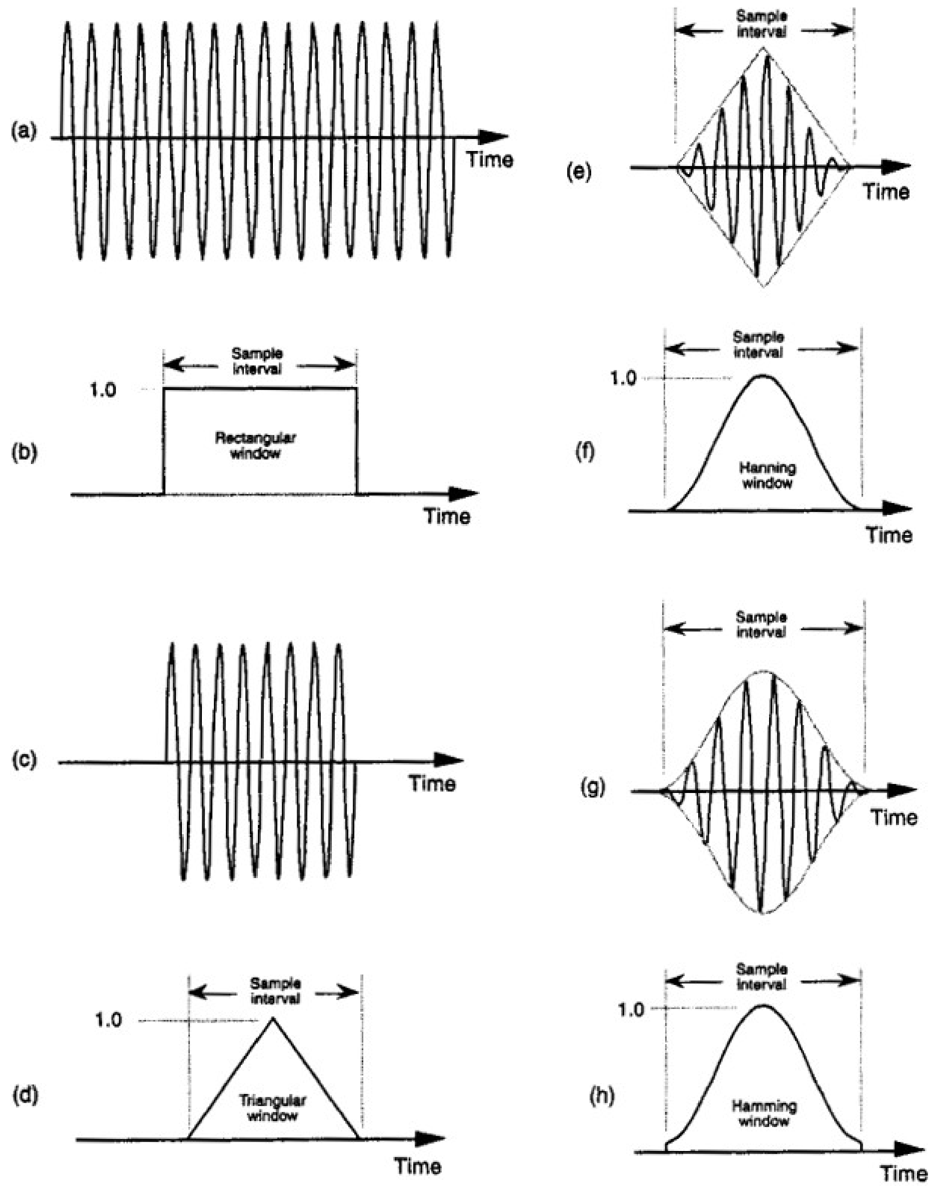
  C: Frequency resolution does not depend on sampling frequency, F_s.

Windowing

- **Leakage, how to reduce it?**

  (how to minimize sample interval end-point discontinuities?)

  - Using windows other than rectangular
    * Triangular
    * Hanning
    * Hamming



 [Lyons]

Windowing

- Coefficients of window functions

Rectangular window $$w(n) = 1, \quad n = 0,1,2,\ldots,L-1$$

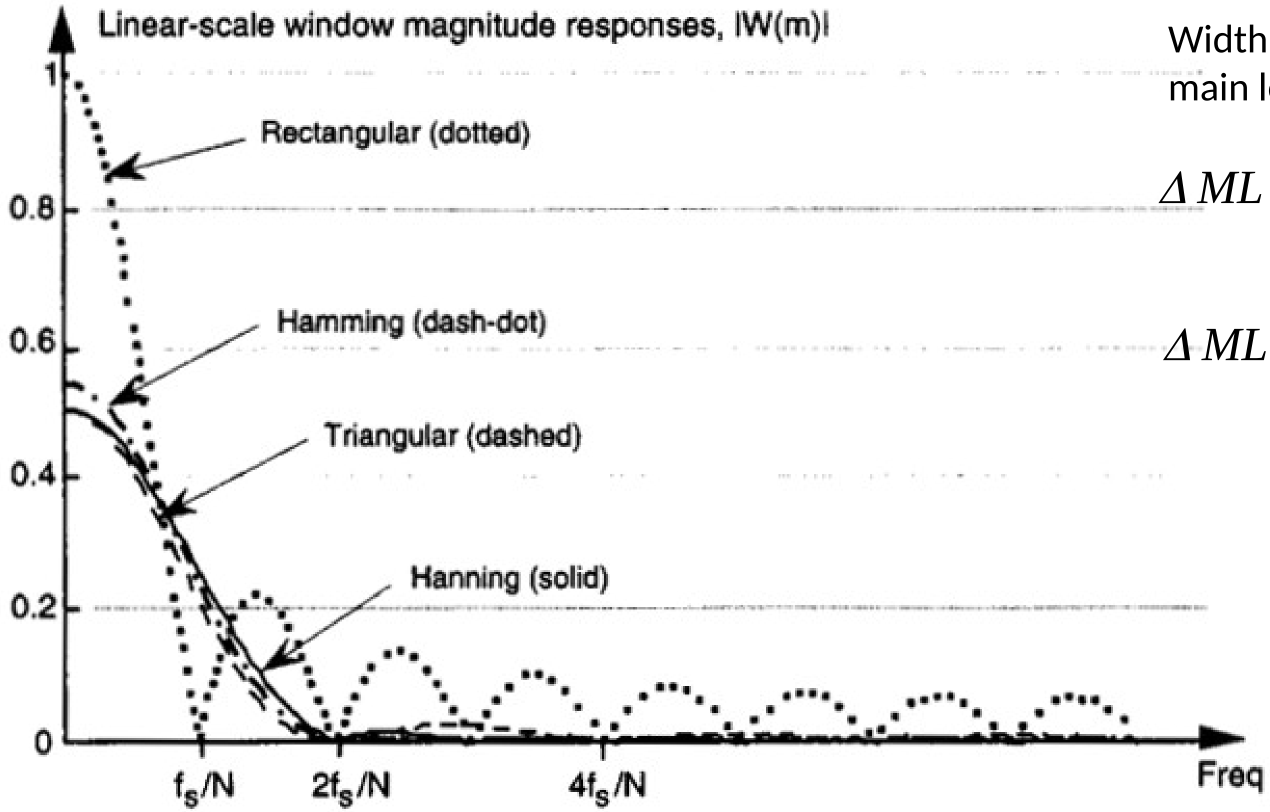
Triangular window $$w(n) = \begin{cases} \dfrac{n}{L/2}, & n = 0,1,2,\ldots,L/2 \\ 2-\dfrac{n}{L/2}, & n = L/2+1, L/2+2,\ldots,L-1 \end{cases}$$

Hanning (Hann) window $$w(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{L}\right), \quad n = 0,1,2,\ldots,L-1$$

Hamming window $$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{L}\right), \quad n = 0,1,2,\ldots,L-1$$

Windowing

- Linear-scale window amplitude spectra, $|W(k)|$ $(N = L)$



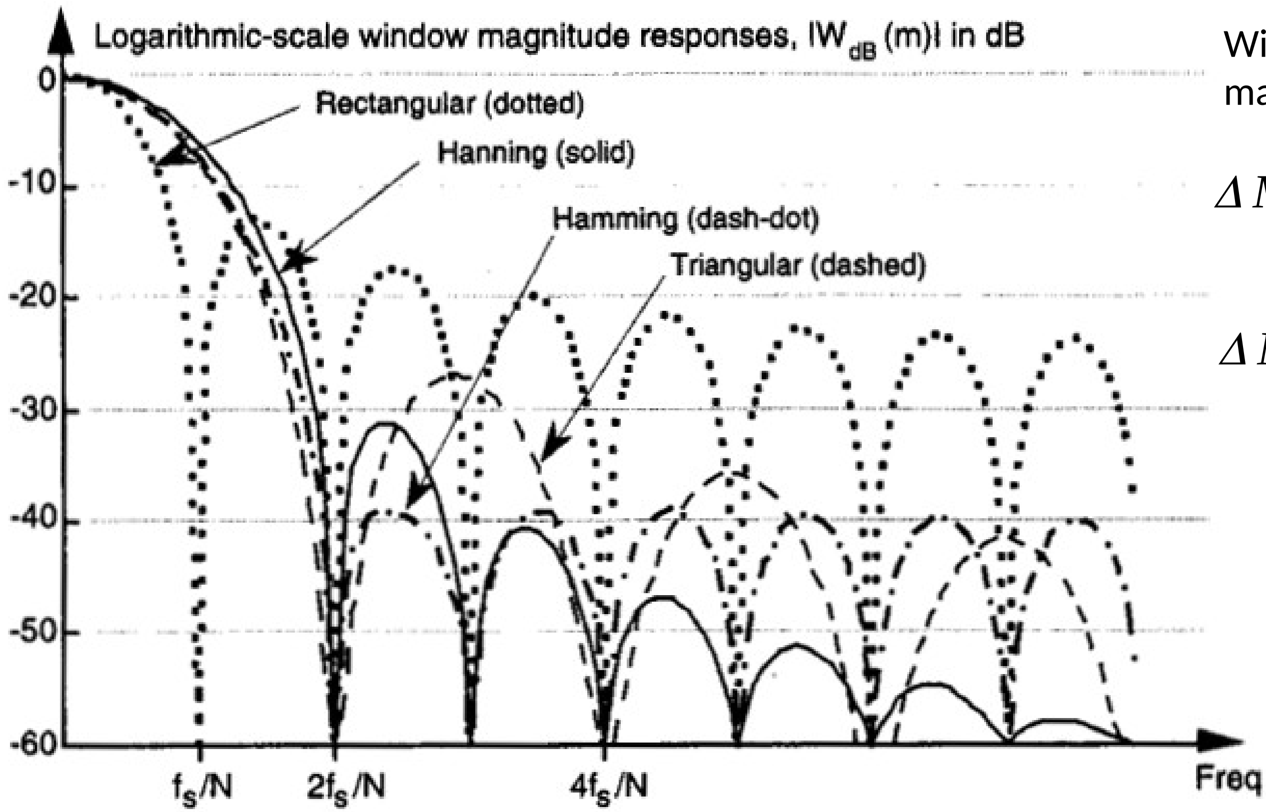
Width of the main lobe

$$\Delta ML_{Rect} = 2.\frac{F_S}{N}$$

$$\Delta ML_{Other} = 4.\frac{F_S}{N}$$

Windowing

- Logarithmic-scale window amplitude spectra, $|W_{dB}(k)|$ in dB (N = L)



Width of the main lobe

$$\Delta ML_{Rect} = 2.\frac{F_S}{N}$$

$$\Delta ML_{Other} = 4.\frac{F_S}{N}$$

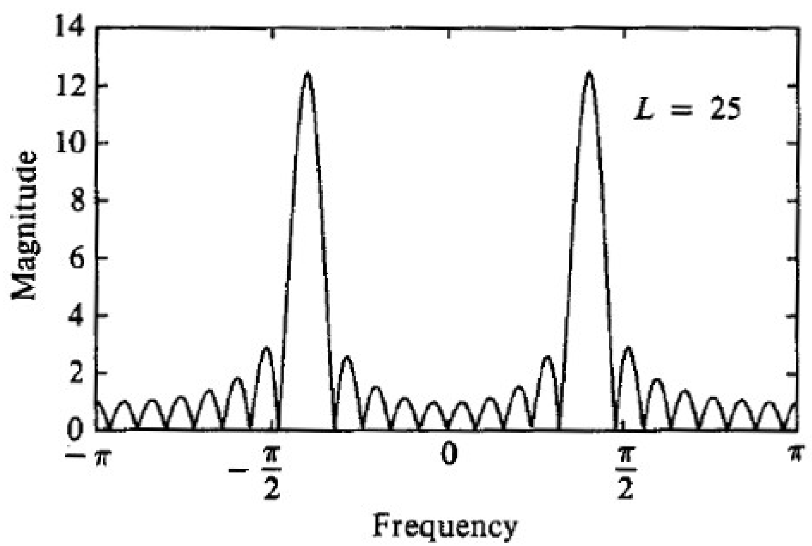
[Lyons]

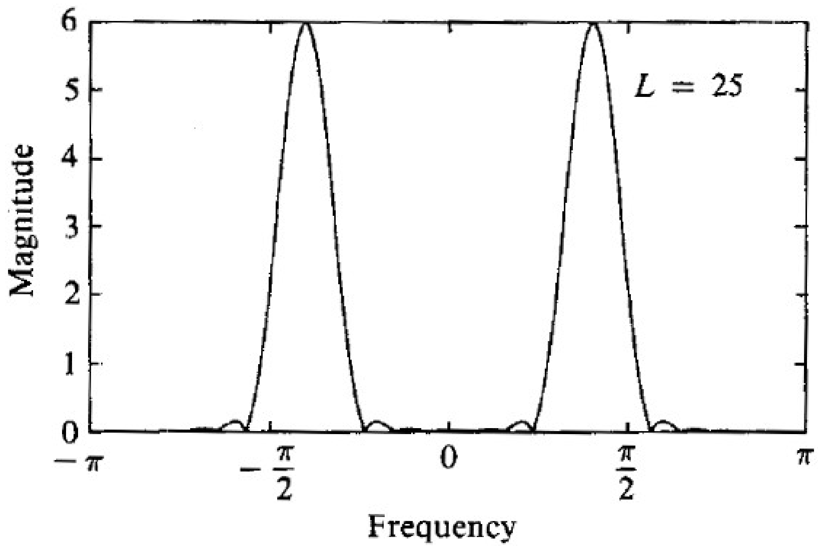
Windowing

- There is a trade-off between the width of the main lobe of the weighting function $|W(k)|$ and the relative amplitude of the main lobe to the side lobes of the weighting function $|W(k)|$
- Resolution is influenced by the width of the main lobe
  - Rectangular window has the narrowest main lobe
    → Higher resolution
  - Other windows have wider main lobes
    → Lower resolution
- Leakage is influenced by the relative amplitude of the main lobe to the side lobes
  - Rectangular window has the highest side lobes relative to the main lobe
    → Higher leakage
  - Other windows have lower side lobes relative to the main lobe
    → Lower leakage

Windowing

- Sinusoid signal, the effect of windowing
  - Left, amplitude spectrum using rectangular window, L = 25 samples
  - Right, amplitude spectrum using Hanning window, L = 25 samples





Width of the main lobe $2.\frac{2.\pi}{L}$

Width of the main lobe $4.\frac{2.\pi}{L}$

Frequency domain analysis of signals - summary

- How to better estimate spectrum using Discrete Fourier Transform?
  - → Sample longer signal segment (higher number of samples)
    - * Using longer weighting function, higher *L* (higher resolution)
    - * Zero padding, *N* - *L* (better interpolation of spectrum only, i.e., higher precision)
- How to improve resolution and reduce leakage?
  - * Frequencies that are close
    - → rectangular window with narrow main lobe (higher resolution)
  - * Frequencies that are apart
    - → Hamming window with lower but constant side lobes (lower leakage close to the frequencies of interest)
    - → Hanning window with side lobes dropping asymptotically (lower leakage far from the frequencies of interest)
- How to improve the signal to noise ratio (SNR)?
  - → Average many successive spectra

Frequency domain analysis of signals

- Example of exam task

Is it possible to detect (to distinguish) the two frequencies of the following signal:

$$x(n) = \sin(\pi\ 0.2\ n) + \sin(\pi\ 0.22\ n)$$

if prior to calculation of the Discrete Fourier Transform (DFT) the rectangular window of length N = 100 samples was used?

A: Yes, it is possible.

B: No, it is not possible, otherwise it is, if we use one of the advanced windows, for example, Hanning window.

C: No, it is not possible at all.

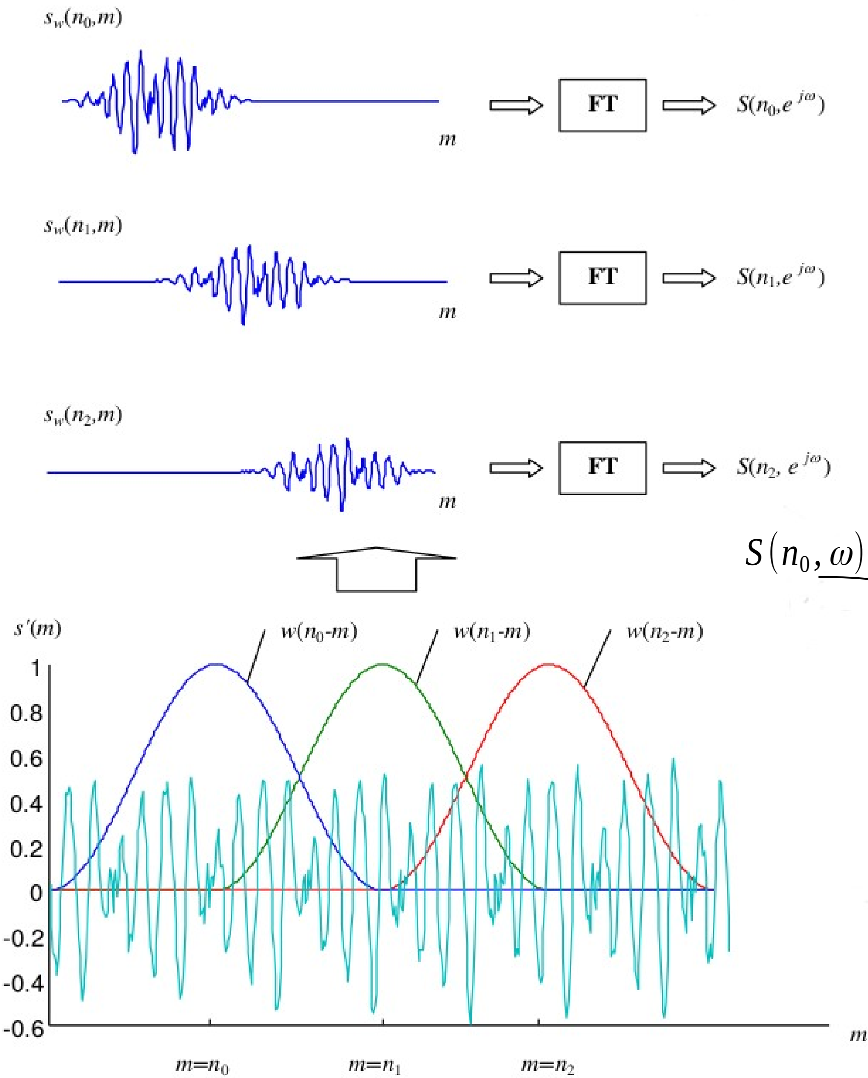
Joint time-frequency analysis

- Amplitude spectrum of a signal only reflects which frequencies exist during the total observation interval, because the Fourier transform integrates frequency components over the total observation interval
- Stationary signals
  - Their frequency content does not change over time
  - Amplitude spectrum is a suitable representation
- Non-stationary signals
  - Their frequency content does change over time
  - Frequent changes are bringing important information (speech, music, other signals, ...)
  - It is important to provide information on when in time different frequencies of a signal occur

  → Perform frequency analysis on short signal segments and move over signal

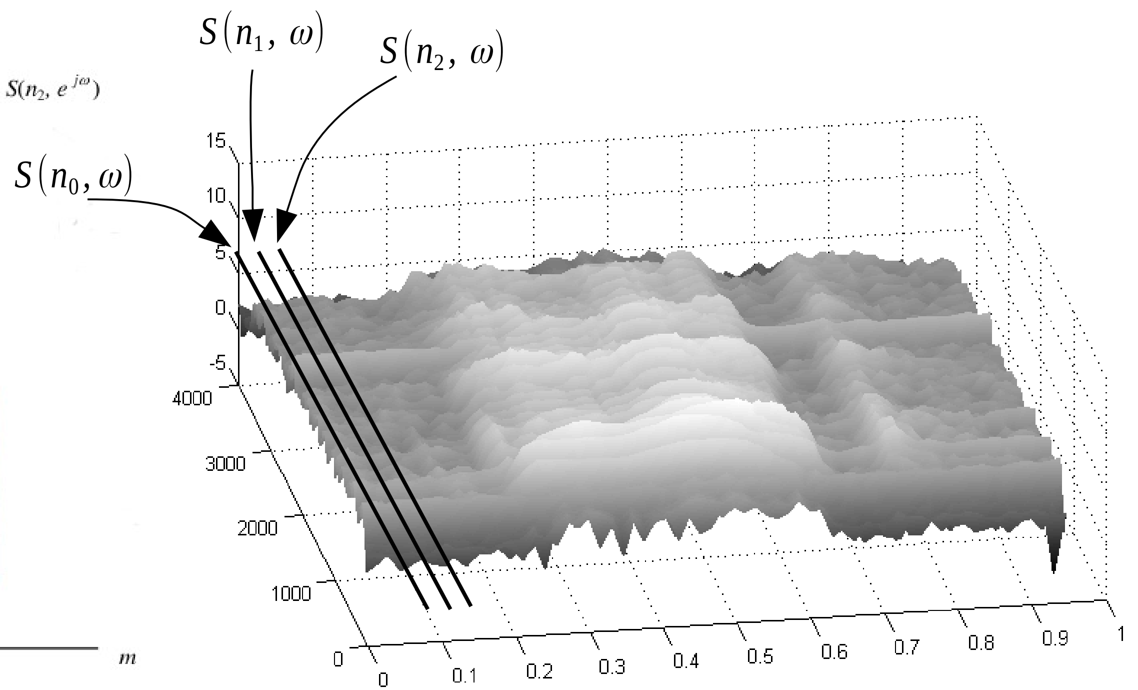
  = Short Term Fourier Transform (STFT) - spectrogram

Short Term Fourier Transform (STFT) -> spectrogram



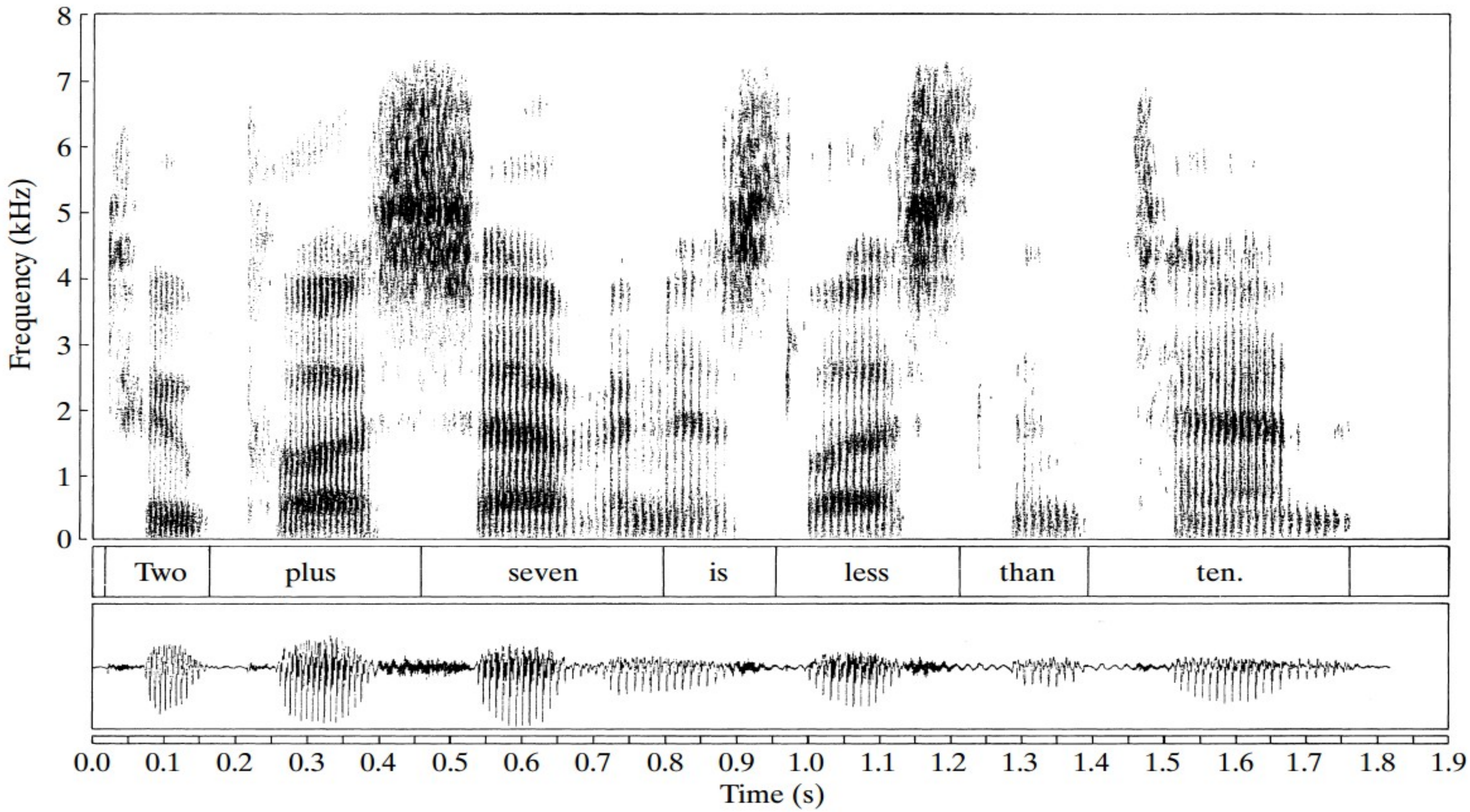
- Divide a signal into short segments
- Calculate amplitude spectrum for each segment
- Compose time series of squared spectra (power spectra)

  → Spectrogram
- Spectrogram of word "devet"



Short Term Fourier Transform (STFT) - spectrogram

- Spectrogram of the speech signal (*F*s = 16.000 smp/sec; Hamming window, *w*(*n*), of duration 6.8 sec, or, *L* = 108; time increment 16 samples, or 1ms)



- For a demo see: https://engineering.purdue.edu/VISE/ee438/demos/

[Oppenheim, Schafer]

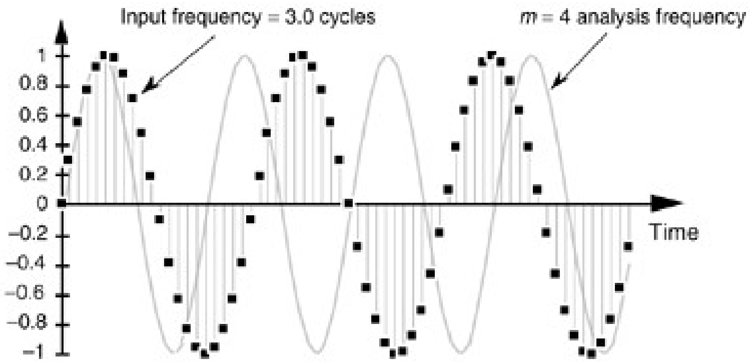
(Additional materials)

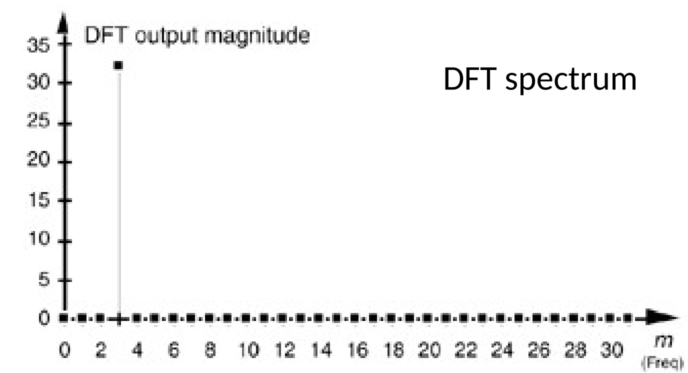
- Frequency domain analysis of signals
- Windowing
- DTFT of windowed (rectangular window) sinusoid signal
- Improving Signal to Noise Ratio (SNR)

Frequency domain analysis of signals

- What actually is going on?

- Ideal case (L = 64)

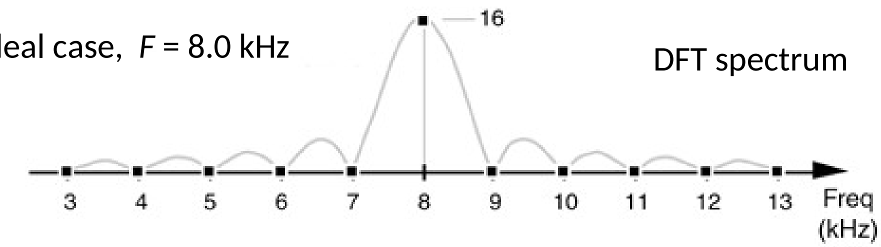




DFT spectrum

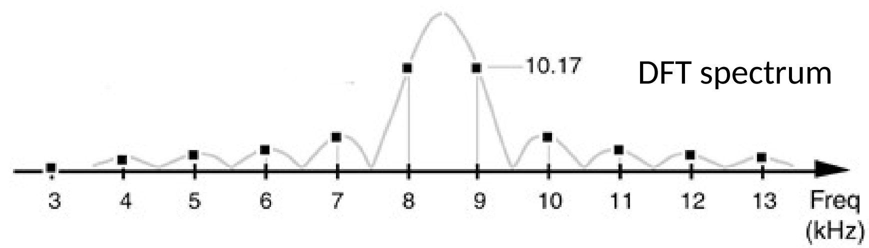
- Examples (Fs = 32 k smp/sec, L = 32)

Ideal case, F = 8.0 kHz



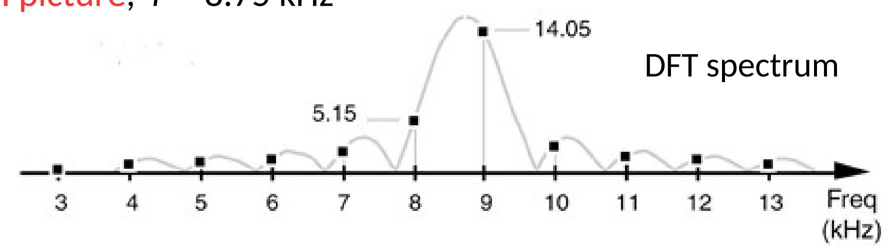
DFT spectrum

Real picture, F = 8.5 kHz



DFT spectrum

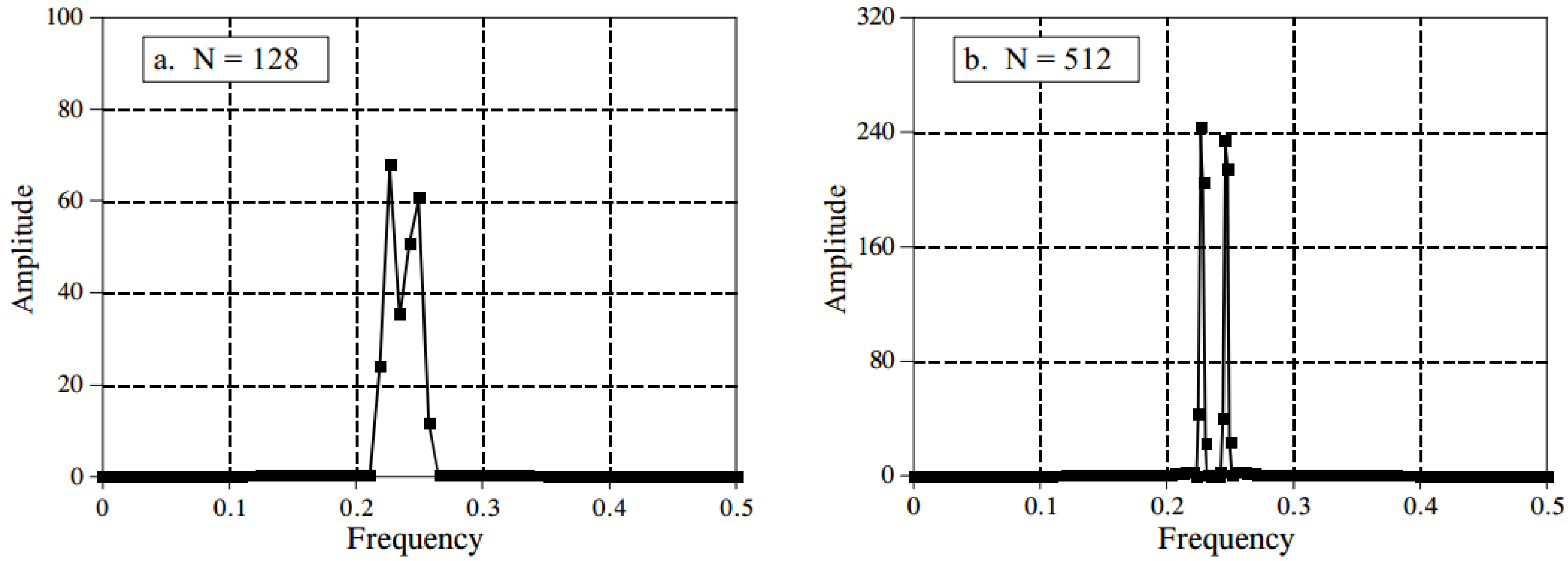
Real picture, F = 8.75 kHz



DFT spectrum

Frequency domain analysis of signals

- Frequency spectrum resolution, how to improve it?
  - The longer the DFT, the better ability to separate closely spaced features
  - A 128 point DFT cannot resolve the two peaks, a 512 point DFT can
    (L = N)

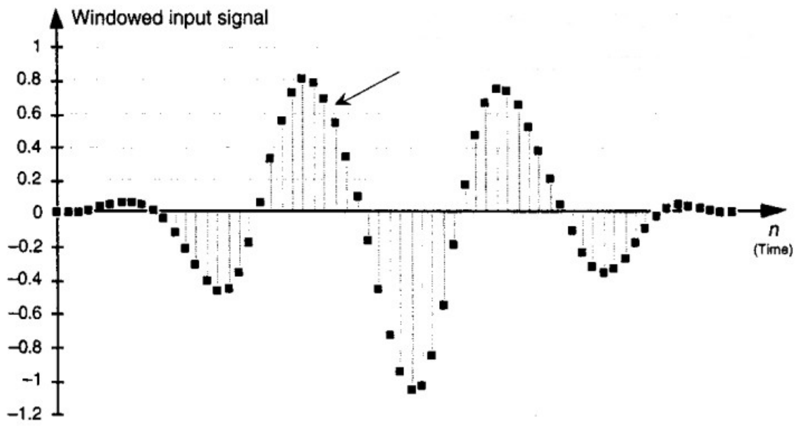


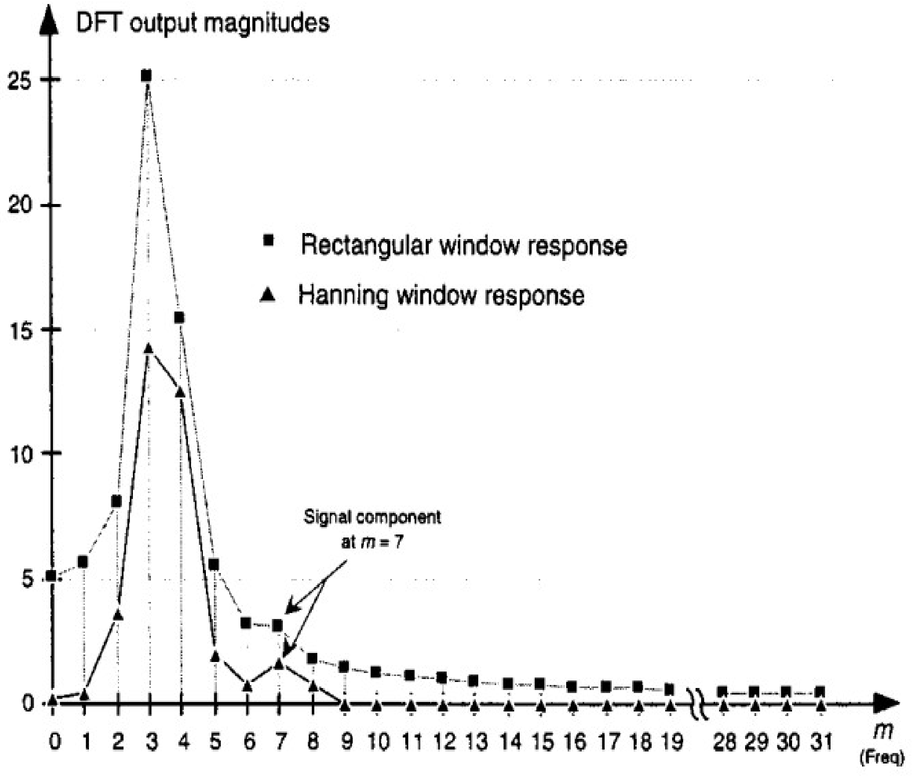
Windowing

- Example of effect using Hanning window

  - Lower leakage, lower resolution

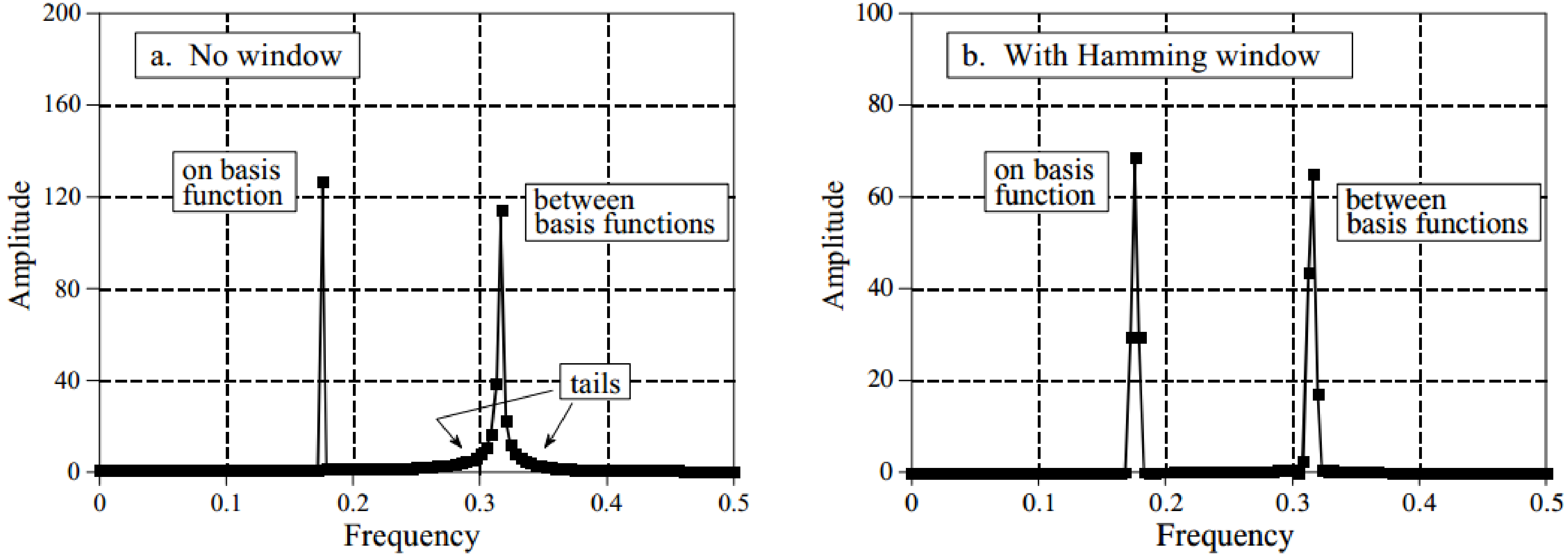
$$x(n) = (0.5 - 0.5\cos(\frac{2\pi n}{64}))\,.\,(\sin(2\pi\, 3.4\, n/64) + 0.1\,\sin(2\pi\, 7\, n/64))$$





Windowing

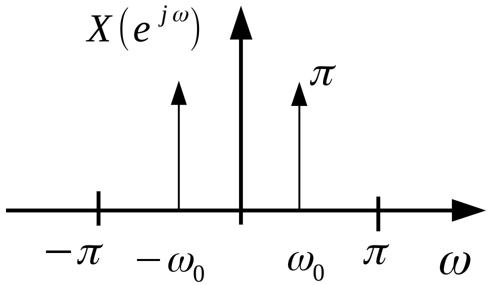
- Example using Hamming window
  - The second sine wave has a frequency between two of the basis functions
  - Lower leakage



DTFT of windowed (rectangular window) sinusoid signal

$$x(n) = \cos(\omega_0 n) = \frac{1}{2} e^{-j\omega_0 n} + \frac{1}{2} e^{j\omega_0 n}$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \pi \left[\delta(\omega - \omega_0 + 2\pi k) + \delta(\omega + \omega_0 + 2\pi k) \right] \longrightarrow$$



$$v(n) = x(n) w(n)$$

Rectangular window, *w(n)*

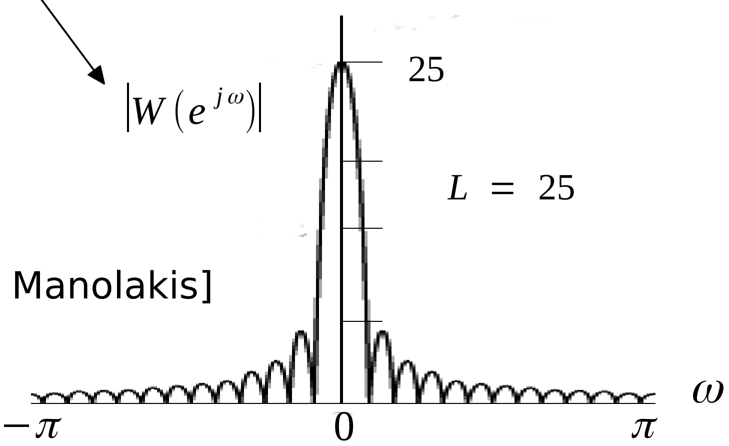
$$w(n) = \begin{cases} 1, & 0 < n < L-1 \\ 0, & \text{otherwise} \end{cases}$$

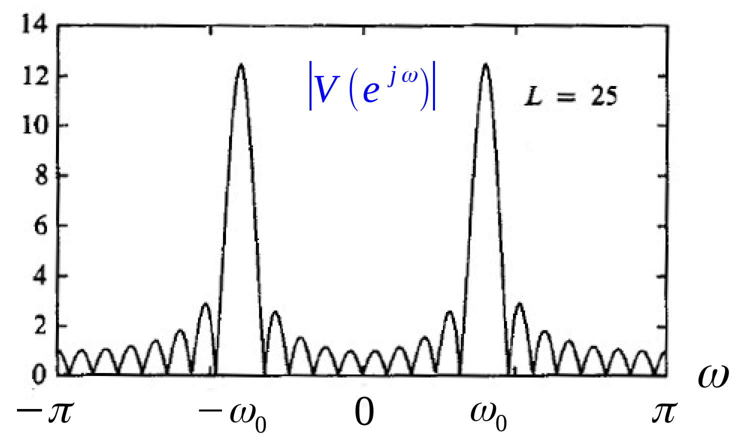
$$W(e^{j\omega}) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

Convolution in frequency domain

$$V(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta$$

$$V(e^{j\omega}) = \frac{1}{2} W(e^{j(\omega+\omega_0)}) + \frac{1}{2} W(e^{j(\omega-\omega_0)})$$





[Proakis, Manolakis]

Improving Signal to Noise Ratio (SNR)

- Example
  - (a) A signal (simulated) measured in the sea (160 smp/sec), noise, but could contain periodic signal (e.g., submarine propeller, 13 Hz and harmonics)
  - (e) Average of 100 spectra obtained through steps from (a) to (d)