The Z transform

- The Z Transform (ZT)
- Examples of Z transform
- The Z transform properties
- The Z transform of a difference equation
- The Inverse Z Transform (IZT)
- Convolution theorem for Z transform
- The Z transform of general Linear Constant Coefficient Difference Equation (LCCDE)
- The transfer (system) function
- Output and impulse response of an LTI system in Z domain
- Rational transfer function
- Transfer function and frequency response
- (Additional materials)

The Z Transform (ZT)

- The Z Transform (ZT) is powerful tool for analyzing discrete-time signals, and for analyzing and designing discrete-time LTI systems
- It is generalization of the Fourier transform (Fourier transform does not exists for all signals)
- It is more convenient to use

The Z Transform (ZT)

- The Discrete-Time Fourier Transform (DTFT) definition

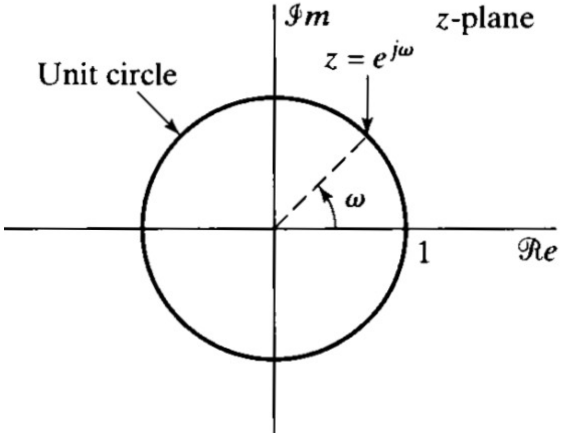
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

- The Z Transform (ZT) is generalization of the DTFT

$$ZT[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

z is complex variable $z = r \,.\, e^{j\omega}$

- If $r = 1, \quad z = e^{j\omega} \rightarrow \text{DTFT}$



- The DTFT is the Z transform on the unit circle

$$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega} \text{ or } |z|=1}$$

- Compare the ZT to DTFT definition

[Oppenheim, Schafer]

Examples of Z transform

- Determine the Z transform of the following signals

$$ZT[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- *x(n)* = {1, 2, 5, 7, 0, 1}

$$\underset{\uparrow\, n=0}{1}$$

$$ZT[x(n)] = X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

- *δ(n)*

$$\delta(n) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$

$$ZT[\delta(n)] = \delta(0) z^{-0} = 1$$

$$\delta(n) \leftrightarrow 1$$

→ Z transform pair

Examples of Z transform

- Derive the Z transform of the following signal (infinite, causal)

$$x(n) = u(n)$$

$$u(n)=\begin{cases}1, & n \geq 0\\ 0, & n < 0\end{cases}$$

$$ZT[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = u(0) + u(1)z^{-1} + u(2)z^{-2} + \ldots + u(n)z^{-n} + \ldots$$

$$X(z) = 1 + z^{-1} + z^{-2} + \ldots + z^{-n} + \ldots$$

$$u(n) \leftrightarrow \frac{1}{1 - z^{-1}}$$

→ Z transform pair

Examples of Z transform

- Derive the Z transform of the following signal (infinite, causal)

$$x(n) = a^n u(n) \qquad a^n u(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad ZT[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = 1u(0) + a^1 u(1) z^{-1} + a^2 u(2) z^{-2} + \ldots + a^n u(n) z^{-n} + \ldots$$

$$X(z) = 1 + a^1 z^{-1} + a^2 z^{-2} + \ldots + a^n z^{-n} + \ldots$$

$$a^n u(n) \leftrightarrow \frac{1}{1 - a z^{-1}}$$

→ Z transform pair

The Z transform properties

- If $x(n) \leftrightarrow X(z)$

Linearity $a_1 x_1(n) + a_2 x_2(n) \leftrightarrow a_1 X_1(z) + a_2 X_2(z)$

Time shift $x(n-n_0) \leftrightarrow z^{-n_0} X(z)$

(Convolution $x_1(n) * x_2(n) \leftrightarrow X_1(z) . X_2(z)$)

The Z transform of a difference equation

- Using linearity and translation property of the Z transform derive the Z transform of the following difference equation

$$y(n) = 2y(n-2) + x(n)$$

$$x(n) \leftrightarrow X(z)$$

$$y(n) \leftrightarrow Y(z)$$

Using $a_1 x_1(n) + a_2 x_2(n) \leftrightarrow a_1 X_1(z) + a_2 X_2(z)$

$x(n-k) \leftrightarrow z^{-k} X(z)$ follows

$$Y(z) = 2Y(z)z^{-2} + X(z)$$

$$Y(z) - 2Y(z)z^{-2} = X(z)$$

$$Y(z)(1 - 2z^{-2}) = X(z)$$

$$\rightarrow \quad Y(z) = X(z)\frac{1}{1 - 2z^{-2}} \quad \rightarrow \quad \frac{Y(z)}{X(z)} = \frac{1}{1 - 2z^{-2}}$$

The Z transform of a difference equation

- We showed how to find the Z transform of the system's output when the system is given as a difference equation
- We related the input and output in another way
- Z transform is one way for solving the difference equations

The Z transform of a difference equation

- Using the properties of the Z transform derive the Z transform of the following difference equation, and then derive shorter expression of the output signal

$$y(n) - 2.5\, y(n-1) \;=\; -y(n-2) + x(n) - 5\, x(n-1) + 6\, x(n-2)$$

- Using the Z transform:

$$\begin{aligned}
Y(z) - 2.5Y(z)z^{-1} \;&=\; -Y(z)z^{-2} + X(z) - 5\,X(z)z^{-1} + 6\,X(z)z^{-2}\\
Y(z) - 2.5Y(z)z^{-1} + Y(z)z^{-2} \;&=\; X(z) - 5\,X(z)z^{-1} + 6\,X(z)z^{-2}\\
Y(z)(1 - 2.5\,z^{-1} + z^{-2}) \;&=\; X(z)(1 - 5\,z^{-1} + 6\,z^{-2})\\
Y(z)(1 - 0.5\,z^{-1})(1 - 2\,z^{-1}) \;&=\; X(z)(1 - 2\,z^{-1})(1 - 3\,z^{-1})\\
Y(z)(1 - 0.5\,z^{-1}) \;&=\; X(z)(1 - 3\,z^{-1})\\
Y(z) \;&=\; 0.5\,Y(z)z^{-1} + X(z) - 3\,X(z)z^{-1}
\end{aligned}$$

- Using the inverse Z transform:

$$y(n) \;=\; 0.5\,y(n-1) + x(n) - 3\,x(n-1)$$

The Inverse Z Transform (IZT)

$$IZT\left[X\left(z\right)\right] = ?$$

- Techniques to derive the inverse Z transform

  - Using the properties of Z transform and Z transform pairs

  - Using the partial fraction expansion method

The Inverse Z Transform (IZT)

- Using the properties of Z transform and Z transform pairs

(properties, linearity, time shift)

$$a_1 x_1(n) + a_2 x_2(n) \leftrightarrow a_1 X_1(z) + a_2 X_2(z)$$

$$x(n-n_0) \leftrightarrow z^{-n_0} X(z)$$

(Z transform pairs)

$$\delta(n) \leftrightarrow 1$$

$$u(n) \leftrightarrow \frac{1}{1-z^{-1}}$$

$$\delta(n-m) \leftrightarrow z^{-m}$$

- Example

$$X(z) = 1 + 3z^{-1} + 2z \quad \rightarrow \quad x(n) = \delta(n) + 3\delta(n-1) + 2\delta(n+1)$$

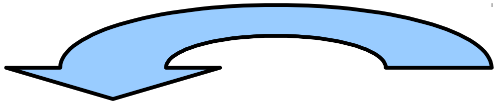
The Inverse Z Transform (IZT)

- Using Z transform properties and pairs

$$a^n u(n) \leftrightarrow \frac{1}{1 - a z^{-1}}$$

$$n\, a^n u(n) \leftrightarrow \frac{a z^{-1}}{(1 - a z^{-1})^2}$$

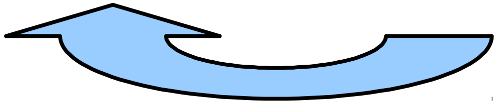
- Example



$$x(n) = 0.5^n u(n) \leftrightarrow \frac{1}{1 - 0.5 z^{-1}} = X(z)$$

- Example (forward, inverse Z transform)

$$x(n) = (1+n)0.5^n u(n) \leftrightarrow \frac{1}{1 - 0.5 z^{-1}} + \frac{0.5 z^{-1}}{(1 - 0.5 z^{-1})^2} = X(z)$$



The Inverse Z Transform (IZT)

- Laboratoy work
  Using the partial fraction expansion method

  \- Example

$$X(z) = \frac{(1+2z^{-1})}{(1+0.4z^{-1}-0.12z^{-2})} = \frac{(1+2z^{-1})}{(1+0.6z^{-1})(1-0.2z^{-1})} = \frac{A_1}{(1+0.6z^{-1})} + \frac{A_2}{(1-0.2z^{-1})}$$

$$X(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{(1-d_1z^{-1})(1-d_2z^{-1})\ \ldots} = \frac{A_1}{1-d_1z^{-1}} + \frac{A_2}{1-d_2z^{-1}} \ldots = \sum_{k=1}^{N} \frac{A_k}{1-d_kz^{-1}}$$

$$A_k = (1-d_kz^{-1})X(z)\Big|_{z=d_k}$$

Convolution theorem for Z transform

Convolution $x_1(n) * x_2(n) \leftrightarrow X_1(z) \,.\, X_2(z)$

- Example, two signals to be convolved, *x1(n)* = {**1**, 2, 2, 1}, *x2(n)* = {**1**, -1} using the Z transform. Verify the result via convolution in time domain.

$$X_1(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3} \qquad X_2(z) = 1 - z^{-1}$$

$$Y(z) = X_1(z)X_2(z) = (1 + 2z^{-1} + 2z^{-2} + z^{-3})(1 - z^{-1})$$

$$Y(z) = 1 + z^{-1} - z^{-3} - z^{-4}$$

Since both signals are of finite length:

$$y(n) = \delta(n) + \delta(n-1) - \delta(n-3) - \delta(n-4)$$

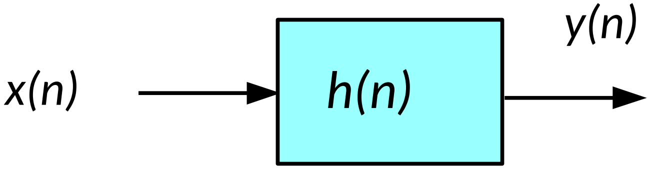
$$x_1(n) = \delta(n) + 2\delta(n-1) + 2\delta(n-2) + \delta(n-3) \qquad x_2(n) = \delta(n) - \delta(n-1)$$

$$x_1(n) * x_2(n) = \delta(n) + \delta(n-1) - \delta(n-3) - \delta(n-4)$$

Convolution theorem for Z transform

Convolution $x_1(n) * x_2(n) \leftrightarrow X_1(z) . X_2(z)$

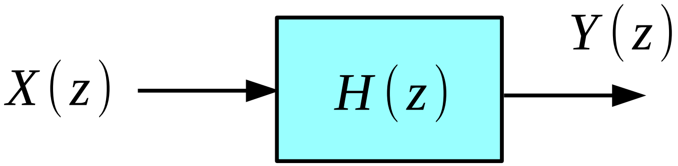
- **Response of an LTI system in time domain (convolution)**



$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

- Convolution in time domain becomes multiplication in Z domain

Output of an LTI system in Z domain



$$Y(z) = ZT[y(n)]$$

$$Y(z) = H(z) . X(z)$$

$$X(z) = ZT[x(n)] \quad H(z) = ZT[h(n)] \quad y(n) = IZT[Y(z)]$$

The Z transform of general Linear Constant-Coefficient Difference Equation (LCCDE)

- We will reduce our focus to those LTI systems which are described by Linear Constant Coefficient Difference Equations (LCCDEs)

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

→ The Z transform of LCCDE LTI systems will be described by fraction of two polynomials

The Z transform of general Linear Constant-Coefficient Difference Equation (LCCDE)

- Discrete-Time Linear Time-Invariant (LTI) systems are described by Linear Constant-Coefficient Difference Equation (LCCDE)

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

- The Z transform

$$Y(z) = -\sum_{k=1}^{N} a_k\, Y(z) z^{-k} + \sum_{k=0}^{M} b_k\, X(z) z^{-k}$$

$$Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

The Z transform of LCCDE LTI systems is described by fraction of two polynomials

→ RATIONAL TRANSFER FUNCTION

The transfer (system) function

- The Z-transform of these systems follows directly from the definition of *H(z)* as the ratio of the output to the input for complex exponential inputs → RATIONAL TRANSFER FUNCTION

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \rightarrow \text{Transfer}(\text{system})\,\text{function} \quad (\text{Transfer characteristic})$$

- If

$$x(n) = \delta(n) \rightarrow X(z) = ZT[\delta(n)] = 1$$

Since the output, *y(n)* = *h(n)*, is the response to the unit sample, *δ(n)*, follows

$$y(n) = h(n) \rightarrow Y(z) = ZT[h(n)] = H(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{ZT[h(n)]}{ZT[\delta(n)]} = \frac{H(z)}{1}$$

- The impulse response h(n) is the inverse Z transform of the transfer function H(z)

$$\rightarrow h(n) = IZT[H(z)]$$

Output and impulse response of an LTI system in Z domain

- Example of exam task

An LTI system is defined by the following difference equation

$$y(n) = 2.5\,y(n-1) - y(n-2) + x(n) - 5\,x(n-1) + 6\,x(n-2)$$

Using the Z transform, derive the impulse response, *h(n)*, of this system and the output signal, *y(n)*, of this system if the input signal, *x(n)*, is following:

$$x(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

Rational transfer function

- Example

Which are the zeros and poles of the following system? Draw zeros and poles in the complex *z* plane (rational transfer function)

$$H(z) = \frac{0.5(1-z^{-1})}{(1-0.1z^{-1})(1-1.5z^{-1})} = \frac{0.5(1-z^{-1})}{(1-1.6z^{-1}+0.15z^{-2})}$$

$$H(z) = 0.5\frac{z^2}{z^2}\frac{(1-z^{-1})}{(1-0.1z^{-1})(1-1.5z^{-1})} = 0.5z\frac{(z-1)}{(z-0.1)(z-1.5)}$$

Zeros: $z_1 = 0$, $z_2 = 1$

Poles: $p_1 = 0.1$, $p_2 = 1.5$

(Rational transfer function)

- The transfer function (characteristic) of an LTI system represented by LCCDE is a rational function of two polynomials

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

$$H(z) = \frac{z^N}{z^M} \cdot \frac{z^M}{z^N} \frac{(b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M})}{(a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N})} = \frac{b_0}{a_0} z^{N-M} \frac{(z - z_1)(z - z_2) \ldots (z - z_M)}{(z - p_1)(z - p_2) \ldots (z - p_N)}$$

- The *M* complex roots of the nominator of *H(z)* are zeros
- The *N* complex roots of the denominator of *H(z)* are poles

- Due to z^{N-M} If $N > M$: $|N - M|$ zeros at $z = 0$

  If $N < M$: $|N - M|$ poles at $z = 0$

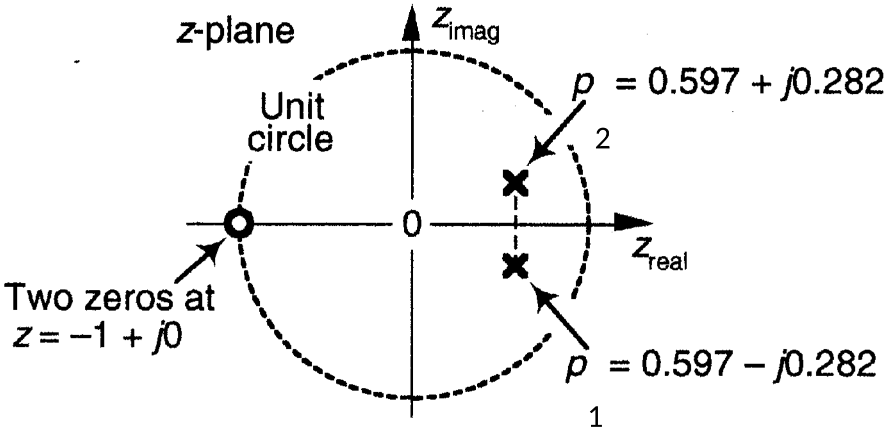
Transfer function and frequency response

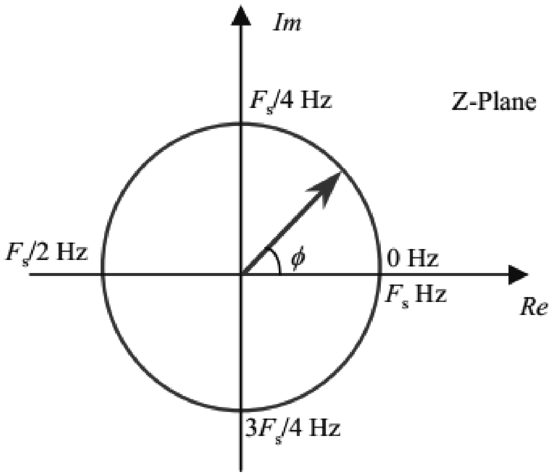
- Transfer function, *H(z)*, is complex function of a complex variable over the entire *Z* plane
- Frequency Response is transfer Function, *H(z)*, evaluated on the unit circle

$$H\left(e^{j\omega}\right) = H(z)\big|_{z=e^{j\omega} \text{ or } |z|=1}$$

- Example

$$H(z) = \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = \frac{(z+1)(z+1)}{(z-0.597+j0.282)(z-0.597-j0.282)}$$





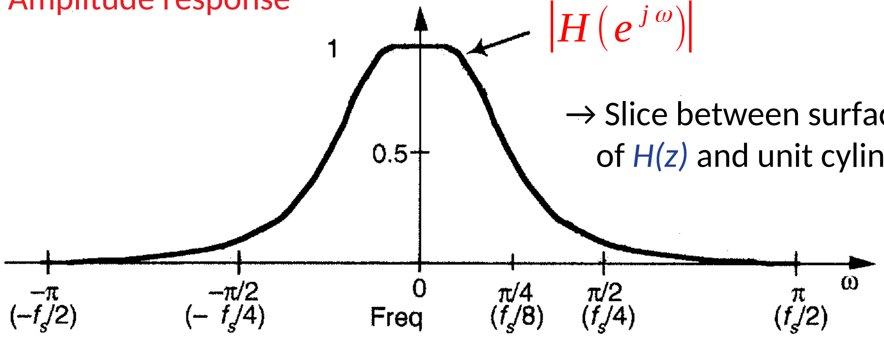
[Lyons]

Transfer function and frequency response

$$H\left(e^{j\omega}\right) = H(z)\big|_{z=e^{j\omega}}$$

$$H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{j\theta(\omega)}$$

Amplitude response

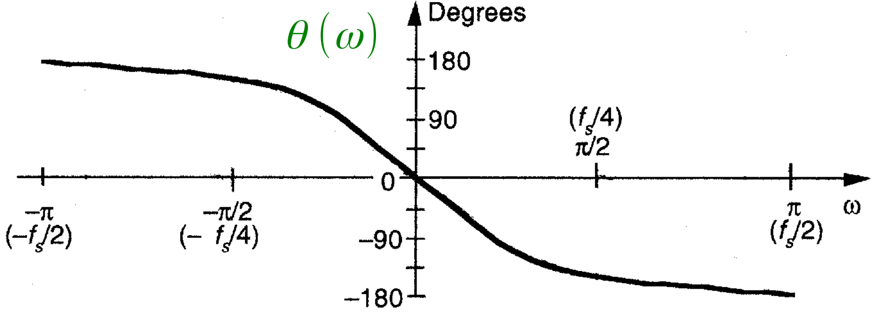


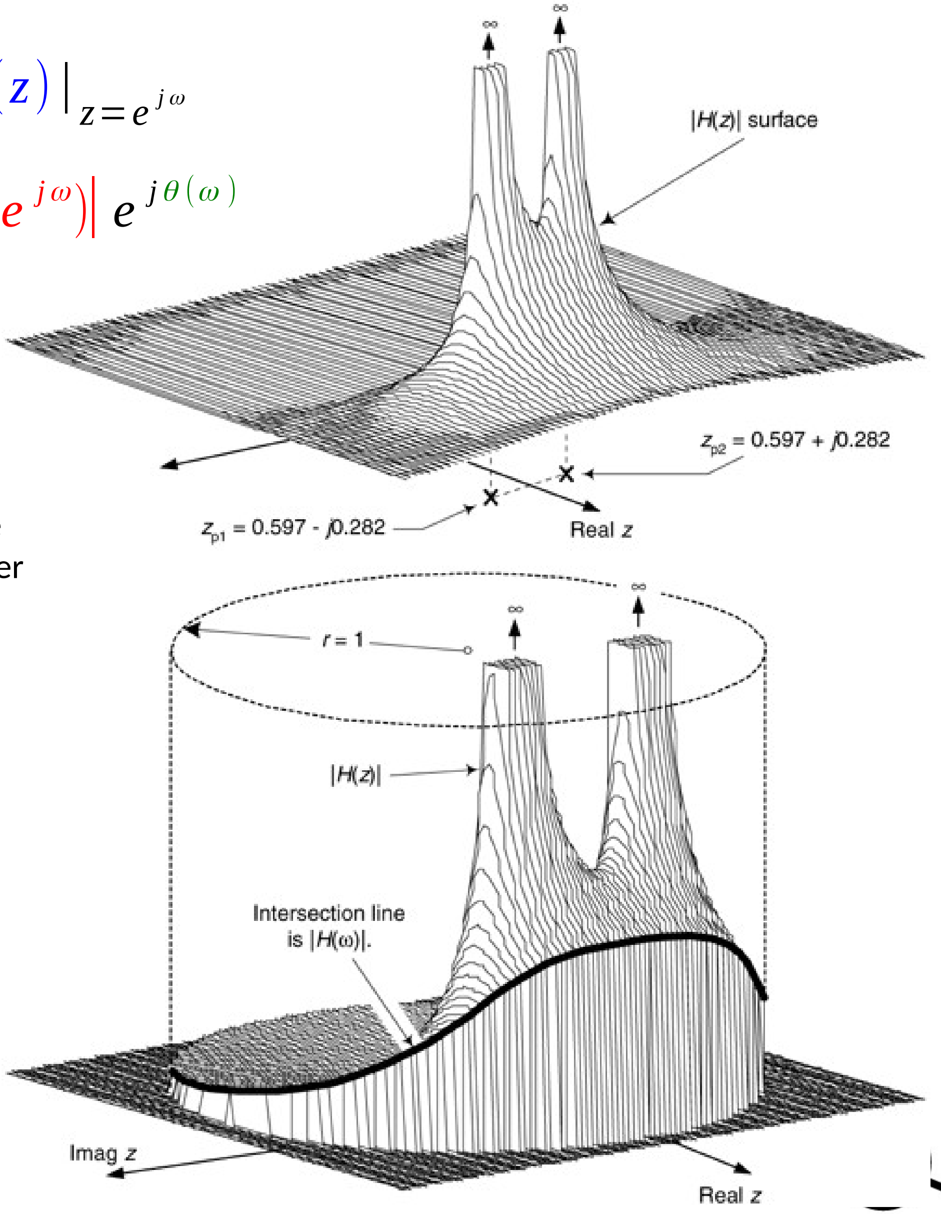
$\left|H\left(e^{j\omega}\right)\right|$

→ Slice between surface of *H(z)* and unit cylinder

Phase response

$\theta(\omega)$





[Lyons]

Transfer function and frequency response

- Example of exam task

What are the zeros and poles of the following system? Sketch zeros and poles in the complex *z* plane

$$H(z) = \frac{2\left(1 - z^{-1}\right)\left(1 + 0.5z^{-1}\right)}{\left(1 - 0.8e^{j\pi/4}z^{-1}\right)\left(1 - 0.8e^{-j\pi/4}z^{-1}\right)}$$

$$H(z) = \frac{z^2}{z^2}\ \frac{2\left(1 - z^{-1}\right)\left(1 + 0.5z^{-1}\right)}{\left(1 - 0.8e^{j\pi/4}z^{-1}\right)\left(1 - 0.8e^{-j\pi/4}z^{-1}\right)}$$

$$H(z) = \frac{2(z-1)(z+0.5)}{\left(z - 0.8e^{j\pi/4}\right)\left(z - 0.8e^{-j\pi/4}\right)}$$

(Additional materials)

- Region of convergence (ROC)
- The inverse Z Transform (IZT)
- The Z transform properties
- The Z transform pairs
- Output and impulse response of an LTI system in Z domain
- Time domain, frequency domain, and Z domain
- Rational transfer function
- Transfer function and frequency response
- Transfer function and frequency response, MATLAB

Region of Convergence (ROC)

- Critical question

  \- Does the sum

$$X(z) = ZT[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

  converge to a finite value?

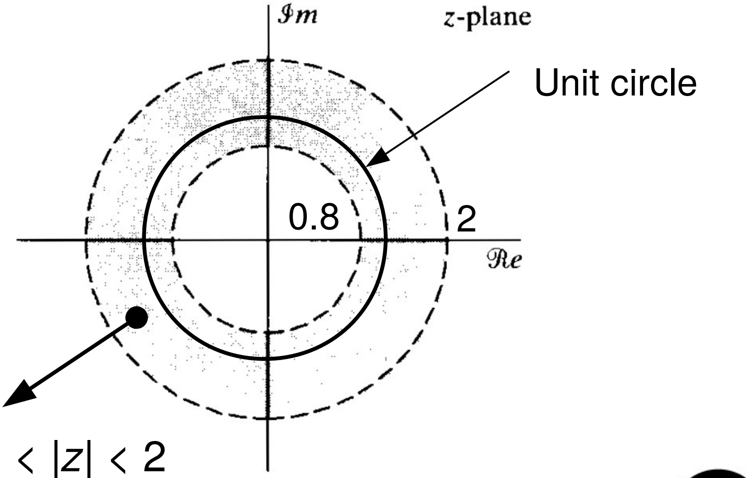
- ROC → Region of complex *z*-plane for which *X(z)* exists (converges, is finite)

$$\longrightarrow \quad \sum_{n=-\infty}^{\infty} |x(n)||z|^{-n} < \infty$$

- Expression for Z transform must specify the ROC

  (Circular regions on the z plane, could be inside circles, outside circles, or in rings)

  (if ROC includes the unit circle, DFTF of *x(n)* exists)



ROC example: 0.8 < |*z*| < 2

[Oppenheim, Schafer]

Region of Convergence (ROC)

- Critical question

  - Does the sum

$$ZT[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

  converge to a finite value?

- ROC (Region Of Convergence)

  → Region of complex *z*-plane for which *X(z)* exists (converges, is finite)

- Expression for Z transform must specify the ROC

Region of Convergence (ROC)

- Determine the Z transform and ROC of the following signals

$$ZT[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- $x(n)$ = {1, 2, 5, 7, 0, 1}
  $\uparrow$ n=0

$$ZT[x(n)] = X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}, \quad \text{for any z, except } z=0$$

- x(n) = {3, 4, 1, 0, 2, 1} (non-causal)
  $\uparrow$ n=0

$$ZT[x(n)] = X(z) = 3z^{2} + 4z^{1} + 1 + 2z^{-2} + 1z^{-3}, \quad \text{any z, except } z=0 \text{ and } z=\infty$$

- $\delta(n)$ $$ZT[\delta(n)] = \delta(0) + \delta(1)z^{-1} + \ldots + \delta(n)z^{-n} + \ldots = 1, \quad \text{for any z}$$

$\delta(n) \leftrightarrow 1$ → Z transform pair

$$\delta(n) = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$$

Region of Convergence (ROC)

- Derive the Z transform and ROC of the following signal (infinite, causal)

$$x(n) = u(n) \qquad u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad ZT[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = u(0) + u(1)z^{-1} + u(2)z^{-2} + \ldots + u(n)z^{-n} + \ldots$$

$$X(z) = 1 + z^{-1} + z^{-2} + \ldots + z^{-n} + \ldots$$

$$u(n) \leftrightarrow \frac{1}{1 - z^{-1}}$$

→ Z transform pair

$$X(z) = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n}$$

$X(z)$ converges to a finite value for $|z^{-1}| < 1$

$$X(z) = \frac{1}{1 - z^{-1}} \quad \rightarrow |z^{-1}| < 1, \text{ or } |z| > 1, \text{ ROC: } |z| > 1$$

Region of Convergence (ROC)

- Derive the Z transform and ROC of the following signal (infinite, causal)

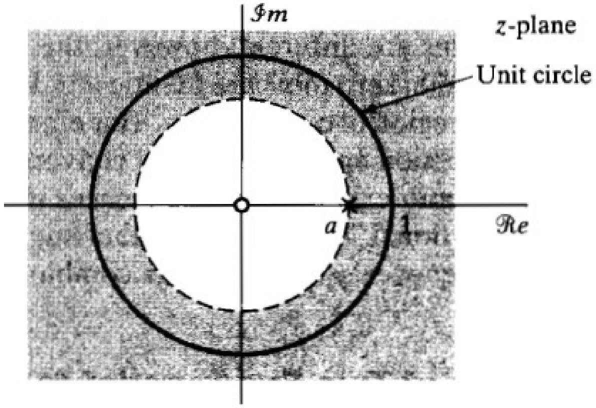
$$x(n) = a^n u(n) \qquad a^n u(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad ZT[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = 1u(0) + a^1 u(1) z^{-1} + a^2 u(2) z^{-2} + \ldots + a^n u(n) z^{-n} + \ldots$$

$$X(z) = 1 + a^1 z^{-1} + a^2 z^{-2} + \ldots + a^n z^{-n} + \ldots$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

$X(z)$ converges to a finite value for $|a z^{-1}| < 1$, or $|z| > |a|$



$$X(z) = \frac{1}{1 - a z^{-1}} \rightarrow |a z^{-1}| < 1, \text{ or } |z| > |a|, \text{ ROC: } |z| > |a|$$

$$a^n u(n) \leftrightarrow \frac{1}{1 - a z^{-1}}$$

→ Z transform pair

If the unit circle lies in the ROC → DTFT exists

The Inverse Z Transform (IZT)

- Techniques to derive the inverse Z transform
  - Using the properties of Z transform
  - Using the partial fraction expansion method
  - Using the table of Z transform pairs

$$IZT[X(z)] = ?$$

$$X(z) = \frac{(1 + 2z^{-1})}{(1 + 0.4z^{-1} - 0.12z^{-2})}$$

- (Definition of the Inverse Z Transform (IZT))

$$x(n) = IZT[X(z)] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\, dz$$

where C represents counterclockwise closed contour at $|z| = r$ within the Region of Convergence (ROC) encircling the origin of the z plane

The Inverse Z Transform (IZT)

$$X(z) = \frac{\left(1 + 2z^{-1}\right)}{\left(1 + 0.4z^{-1} - 0.12z^{-2}\right)}$$

$$IZT\left[X(z)\right] = ?$$

- Techniques to derive the inverse Z transform

  - Using the properties of Z transform and Z transform pairs

  - Using the partial fraction expansion method

The Z transform properties

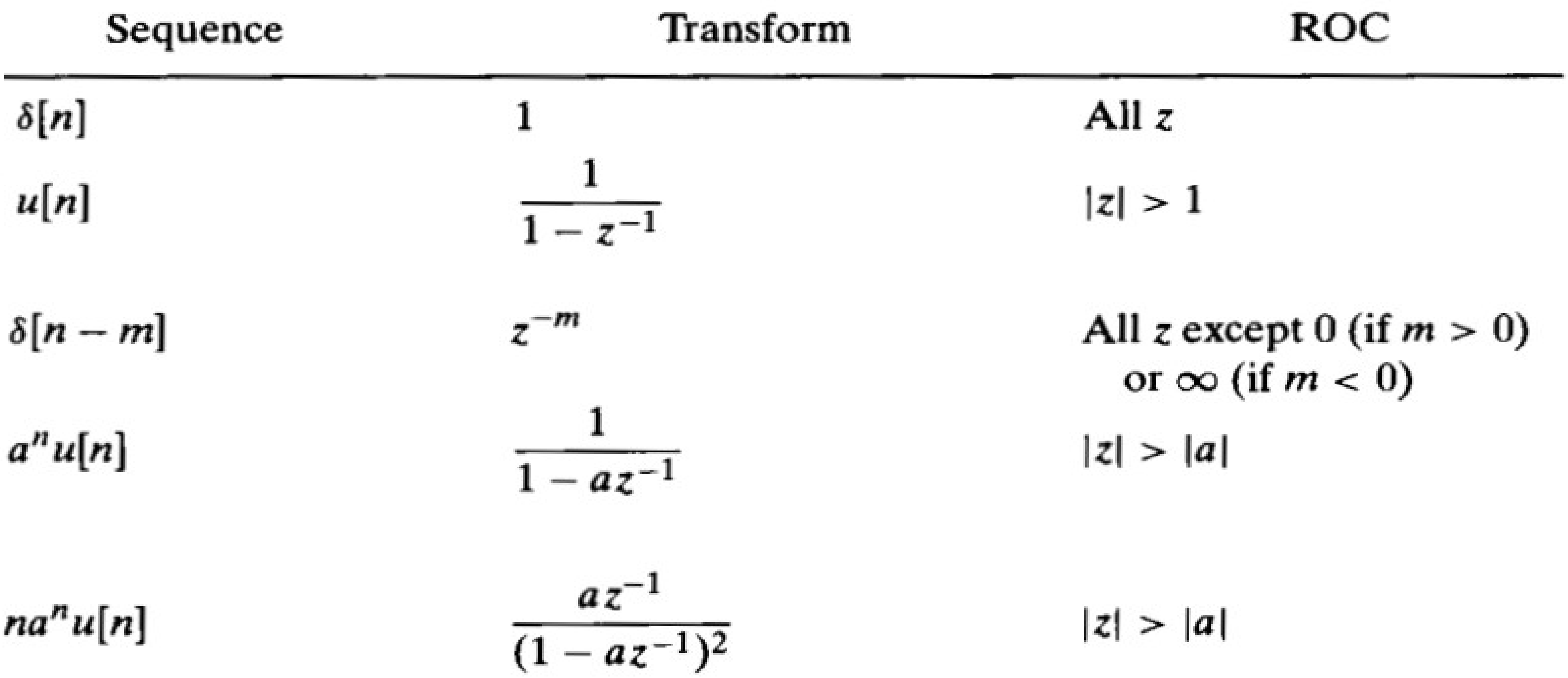
- If $x(n) \leftrightarrow X(z) \quad ROC = R_x$

Linearity $a_1 x_1(n) + a_2 x_2(n) \leftrightarrow a_1 X_1(z) + a_2 X_2(z) \quad R = R_{x1} \cap R_{x2}$

Time shift $x(n-n_0) \leftrightarrow z^{-n_0} X(z) \quad R = R_x$, except
$n_0 > 0:\ z=0$ or
$n_0 < 0:\ z=\infty$

Convolution $x_1(n) * x_2(n) \leftrightarrow X_1(z) \,.\, X_2(z) \quad R = R_{x1} \cap R_{x2}$

The Z transform pairs

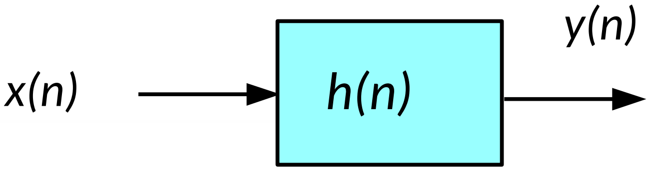


Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\dfrac{1}{1 - z^{-1}}$	$\|z\| > 1$
$\delta[n - m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
$a^n u[n]$	$\dfrac{1}{1 - az^{-1}}$	$\|z\| > \|a\|$
$na^n u[n]$	$\dfrac{az^{-1}}{(1 - az^{-1})^2}$	$\|z\| > \|a\|$

[Oppenheim, Schafer]

Output and impulse response of an LTI system in Z domain

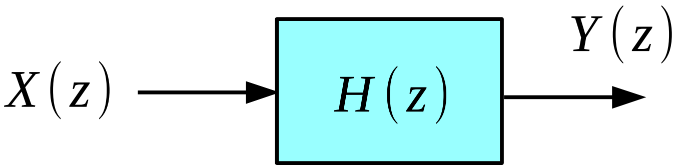
- Response of an LTI system in time domain (convolution)



$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- Convolution in time domain becomes multiplication in Z domain

Output of an LTI system in Z domain



$$X(z) = ZT[x(n)]$$

$$H(z) = ZT[h(n)]$$

$$Y(z) = ZT[y(n)]$$

Task 1: $x(n),\ h(n),\quad y(n) = ?$

1.a $Y(z) = H(z)\ .\ X(z)$

1.b $y(n) = IZT[Y(z)]$

Task 2: $x(n),\ y(n),\quad h(n) = ?$

2.a $H(z) = Y(z)\,/\,X(z)$

2.b $h(n) = IZT[H(z)]$

Output and impulse response of an LTI system in Z domain

- Example (Task 1: *x(n)*, *h(n)*, *y(n)* = ?)

  - An LTI system is defined by the following difference equation and impulse response, *h(n)*

$$y(n) = x(n) - x(n-1)$$

$$h(n) = \{1, -1\} = \delta(n) - \delta(n-1)$$

Determine the output, *y(n)*, of the system to the following input signal, *x(n)*

$$x(n) = \{1, 1, 1\} = \delta(n) + \delta(n-1) + \delta(n-2)$$

$$H(z) = 1 - z^{-1} \qquad X(z) = 1 + z^{-1} + z^{-2} \qquad Y(z) = H(z).X(z) = 1 - z^{-3}$$

$$y(n) = IZT[Y(z)] = \delta(n) - \delta(n-3)$$

Output and impulse response of an LTI system in Z domain

- Example (Task 2: *x(n)*, *y(n)*, *h(n)* = ?)

  - What we know are the input, *x(n)*, and the output, *y(n)*, signals of an LTI system

$$x(n) = \{1, 1, 1\} = \delta(n) + \delta(n-1) + \delta(n-2)$$

$$y(n) = \{1, 0, 0, -1\} = \delta(n) - \delta(n-3)$$

Determine the impulse response, *h(n)*, and difference equation of the system

$$X(z) = 1 + z^{-1} + z^{-2} \qquad Y(z) = 1 - z^{-3}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-3}}{1 + z^{-1} + z^{-2}} = \frac{(1 - z^{-1})(1 + z^{-1} + z^{-2})}{(1 + z^{-1} + z^{-2})} = 1 - z^{-1}$$

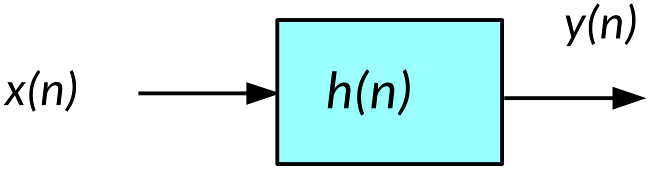
$$h(n) = IZT[H(z)] = \delta(n) - \delta(n-1)$$

$$Y(z) = H(z).X(z) = (1 - z^{-1})X(z) = X(z) - X(z)z^{-1}$$

$$y(n) = IZT[Y(z)] = x(n) - x(n-1)$$

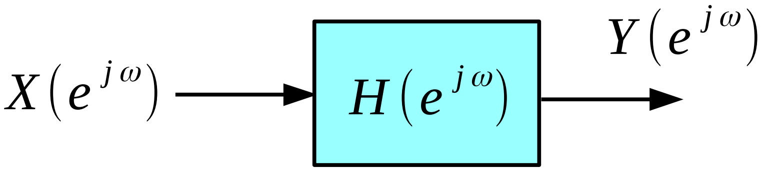
Time domain, frequency domain, and Z domain

- Response of an LTI system in time domain (convolution)



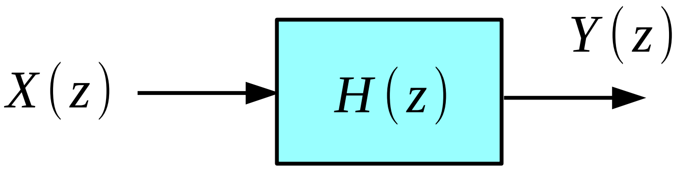
$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- Output of an LTI system in frequency domain (convolution → multiplication)



$$Y(e^{j\omega}) = H(e^{j\omega})\,.\,X(e^{j\omega})$$

- Output of an LTI system in Z domain (convolution → multiplication)



$$Y(z) = H(z)\,.\,X(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$X(z) = ZT[x(n)]$$
$$Y(z) = ZT[y(n)]$$
$$H(z) = ZT[h(n)]$$

- $h(n)$, $H(e^{j\omega})$, $H(z)$ are different external representations of an LTI system

Rational transfer function

- The transfer function (characteristic) of an LTI system represented by LCCDE is a rational function of two polynomials

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

$$H(z) = \frac{z^N}{z^M} \cdot \frac{z^M}{z^N} \frac{(b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M})}{(a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N})} = \frac{b_0}{a_0} z^{N-M} \frac{(z - z_1)(z - z_2) \ldots (z - z_M)}{(z - p_1)(z - p_2) \ldots (z - p_N)}$$

- The *M* complex roots of the nominator of *H(z)* are zeros
- The *N* complex roots of the denominator of *H(z)* are poles

- Due to z^{N-M} If $N > M$: $|N - M|$ zeros at $z = 0$

  If $N < M$: $|N - M|$ poles at $z = 0$

Transfer function and frequency response

- Transfer function, $H(z)$, is complex function of a complex variable over entire z plane

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} z^{N-M} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

- Frequency Response , $H(e^{j\omega})$, is Transfer function evaluated on the unit circle

Frequency response: $$H(e^{j\omega}) = \frac{b_0}{a_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M} (e^{j\omega} - z_k)}{\prod_{k=1}^{N} (e^{j\omega} - p_k)} \quad \longleftarrow \quad z = e^{j\omega}$$

Amplitude response: $$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} |e^{j\omega} - z_k|}{\prod_{k=1}^{N} |e^{j\omega} - p_k|}$$

→ Slice between surface of $H(z)$ and unit cylinder

Phase response: $$\theta(\omega) = \sphericalangle\left\{\frac{b_0}{a_0}\right\} + \omega(N-M) + \sum_{k=1}^{M} \sphericalangle\{e^{j\omega} - z_k\} - \sum_{k=1}^{N} \sphericalangle\{e^{j\omega} - p_k\}$$

Transfer function and frequency response, MATLAB

- MATLAB, deriving zero-pole plot of a transfer function

$$H(z)=\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}=\frac{(z+1)(z+1)}{(z-0.597+j0.282)(z-0.597-j0.282)}$$

- MATLAB

```
>>  P(1,1) = 0.597  - 0.282i;     %  Pole  p1
>>  P(2,1) = 0.597 + 0.282i;      %  Pole  p2
>>  Z(1,1) = -1.0 + 0.0i;         %  Zero  z1
>>  Z(2,1) = -1.0  – 0.0i;        %  Zero  z2
>>  …
>>  zplane(Z, P)                  % Zero-pole plot
>>  …
```