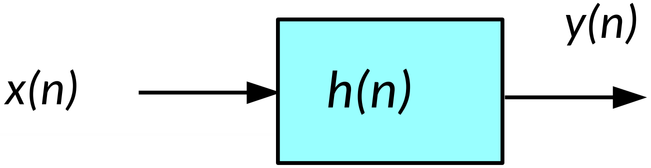
Fourier domain, II

- Determining the output of an LTI system in frequency domain
- Frequency response of LTI system
- Frequency response of LTI system, example
- Frequency response of LTI system
- Frequency response of LTI system, example 2

Determining the output of an LTI system in frequency domain

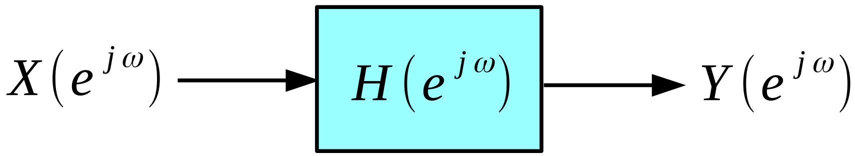
- Response of LTI system in time domain (convolution in time domain)



$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

- Frequency domain relationship for determining the output of an LTI system

→ Convolution in time domain becomes multiplication in frequency domain



Spectrum of the output signal

$$Y(e^{j\omega}) = H(e^{j\omega}) \,.\, X(e^{j\omega})$$

$$X(e^{j\omega}) = \mathrm{DTFT}[x(n)]$$

$$H(e^{j\omega}) = \mathrm{DTFT}[h(n)]$$

$$Y(e^{j\omega}) = \mathrm{DTFT}[y(n)]$$

Output signal

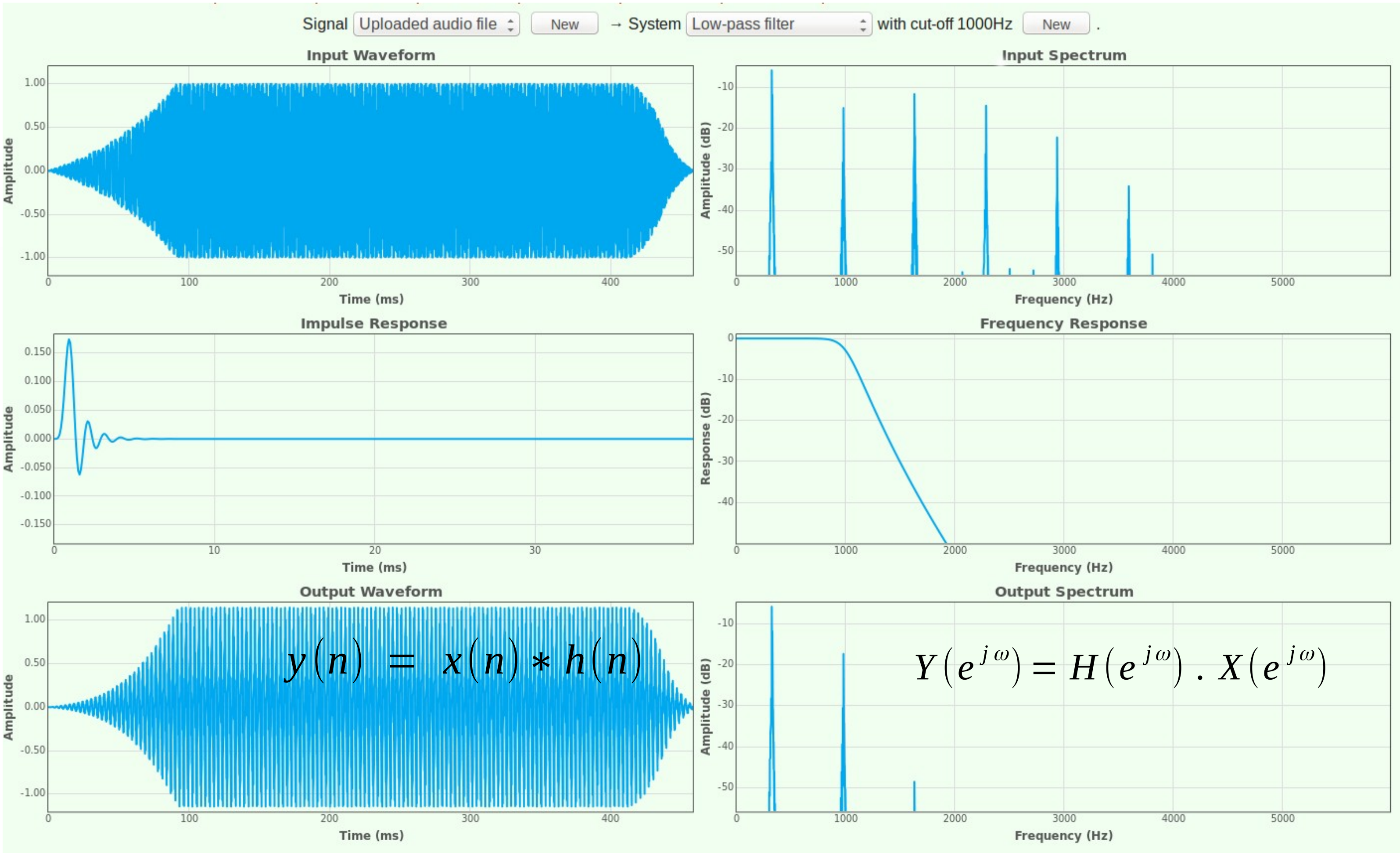
$$y(n) = \mathrm{IDTFT}[\,Y(e^{j\omega})\,]$$

Frequency response of LTI system

$x(n)$ Play

$h(n)$

$y(n)$ Play



$$y(n) = x(n) * h(n)$$

$$Y(e^{j\omega}) = H(e^{j\omega}) \, . \, X(e^{j\omega})$$

$X(e^{j\omega})$

$H(e^{j\omega})$

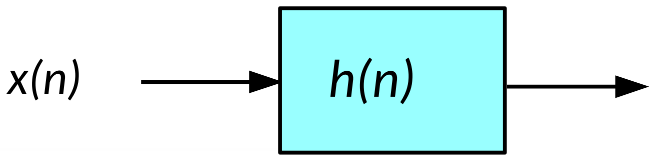
$|H(e^{j\omega})|$

$Y(e^{j\omega})$

- http://www.speechandhearing.net/laboratory/esystem/

Frequency response of LTI system

- Impulse response *h(n)* completely describes LTI system



$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

- Complex sinusoid as input signal

$$x(n) = e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,e^{j\omega(n-k)}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,e^{-j\omega k} \,.\, e^{j\omega n}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)\,e^{-j\omega k}$$

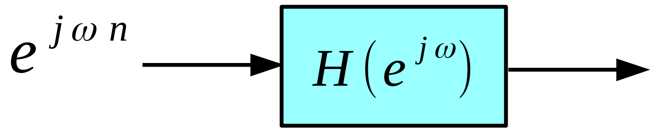
Frequency response of LTI system is the DTFT of the impulse response *h(n)* of the system

$$y(n) = H(e^{j\omega}) \,.\, e^{j\omega n}$$

→ Output is sinusoid scaled by DTFT of *h(n)* at ω

Frequency response of LTI system

- If the input signal is complex sinusoid $x(n) = e^{j\omega n}$



$$y(n) = H(e^{j\omega}) \,.\, e^{j\omega n}$$

- Since *frequency response* is

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j H_I(e^{j\omega})$$

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| \,.\, e^{j\theta(\omega)} \quad \text{(polar notation !)}$$

where $\left|H(e^{j\omega})\right|$ is *amplitude response* (gain) of the system

$\theta(\omega)$ is *phase response* (phase shift) of the system

follows: $y(n) = \left|H(e^{j\omega})\right| \,.\, e^{j(\omega n + \theta(\omega))}$

The output of the system with impulse response *h(n)* is the same sinusoid

scaled by $\left|H(e^{j\omega})\right|$ and phase-shifted by $\theta(\omega)$

where $H(e^{j\omega}) = \mathrm{DTFT}[h(n)]$

Frequency response of LTI system

- In practice signals, *x(n)*, *h(n)*, are real
- Example

$$x(n) = A\cos(\omega_0 n + \Phi)$$

(Derivation)

$$x(n) = \frac{A}{2}\left(e^{j(\omega_0 n + \Phi)} + e^{-j(\omega_0 n + \Phi)}\right)$$

$$x(n) = \frac{A}{2} e^{j\Phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\Phi} e^{-j\omega_0 n}$$

$$y(n) = \frac{A}{2} e^{j\Phi} H(e^{j\omega_0})\, e^{j\omega_0 n} + \frac{A}{2} e^{-j\Phi} H(e^{-j\omega_0})\, e^{-j\omega_0 n}$$

$$\text{For real } h(n) \rightarrow H(e^{-j\omega}) = H^*(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{-j\Theta(\omega)}$$

Output:

$$y(n) = A\left|H(e^{j\omega_0})\right| \cos(\omega_0 n + \Phi + \theta(\omega_0))$$

Gain → $\left|H(e^{j\omega_0})\right|$

Phase shift → $\theta(\omega_0)$

Frequency response of LTI system

- Passing a real sinusoid through an LTI system with real *h(n)*

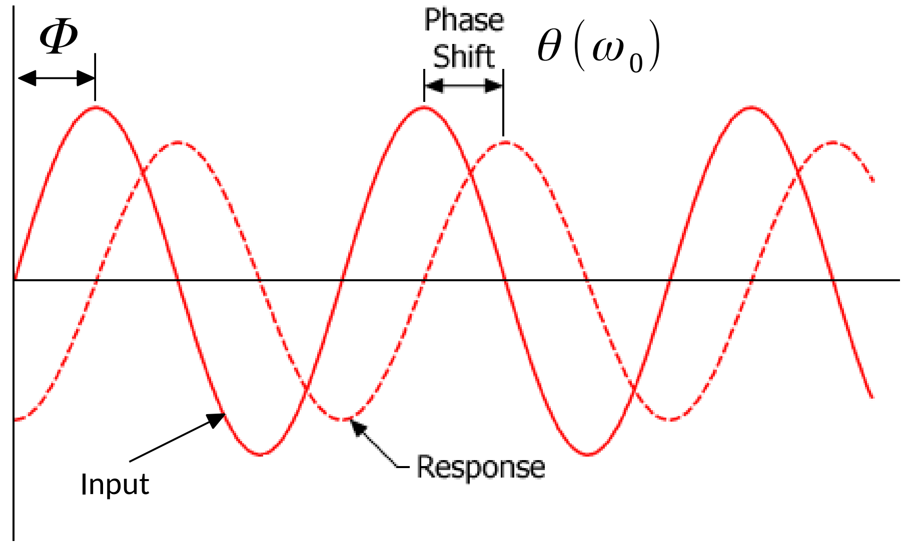
Input

$$A\cos\left(\omega_0 n + \Phi\right) \rightarrow \boxed{H\left(e^{j\omega}\right)} \rightarrow$$

Response

$$A\left|H\left(e^{j\omega_0}\right)\right|\cos\left(\omega_0 n + \Phi + \theta\left(\omega_0\right)\right)$$

- Gain is modified by $\left|H\left(e^{j\omega_0}\right)\right|$
- Phase is shifted by $\theta\left(\omega_0\right)$



Frequency response of LTI system

- LTI system

$$x(n) = e^{j\omega n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y(n) = H(e^{j\omega}) \,.\, e^{j\omega n}$$

- Arbitrary input signal as a sum of sinusoids

$$x(n) = \sum_{i=1}^{L} A_i \cos(\omega_i n + \Phi_i)$$

- The output signal is the sum of responses

$$y(n) = \sum_{i=1}^{L} A_i \left| H(e^{j\omega_i}) \right| \cos(\omega_i n + \Phi_i + \theta(\omega_i))$$

Influence of the system to the *i*-th component of the signal

Frequency response of LTI system

- LTI system

$$x(n) = e^{j\omega n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y(n) = H(e^{j\omega}) \,.\, e^{j\omega n}$$

- Frequency response

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j H_I(e^{j\omega})$$

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| \,.\, e^{j\theta(\omega)}$$

- Amplitude response (gain)

$$\left|H(e^{j\omega})\right| = \sqrt{H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})}$$

- Phase response (phase shift)

$$\theta(\omega) = \sphericalangle H(e^{j\omega}) = \arctan\left(\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right)$$

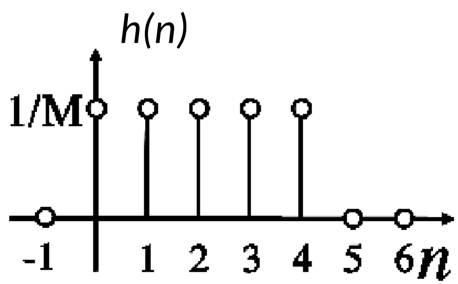
Frequency response of LTI system, example

- Derive frequency response, amplitude response, and phase response of the following moving average filter

$$y(n) = \frac{1}{M}\sum_{l=0}^{M-1} x(n-l), \quad M = 5$$

$$h(n) = \{\mathbf{1/5},\ 1/5,\ 1/5,\ 1/5,\ 1/5\}$$

$$h(n) = \frac{1}{M}\sum_{l=0}^{M-1} \delta(n-l)$$



- Study the output *y(n)* for the following cases: $\omega_0 = 0.1\,\pi$ $\omega_1 = 0.5\,\pi$

Frequency response of LTI system, example

- Frequency response of moving average filter

$$H(e^{j\omega}) = \mathrm{DTFT}[h(n)] = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} = \frac{1}{M}\sum_{n=0}^{M-1} e^{-j\omega n}$$

$$H(e^{j\omega}) = \frac{1}{M}\left(1+e^{-j\omega}+e^{-j\omega 2}+\ldots+e^{-j\omega(M-1)}\right) = \frac{1}{M}\,\frac{1-e^{-j\omega M}}{1-e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1}{M} e^{-j\omega\frac{(M-1)}{2}} \frac{\sin(M\omega/2)}{\sin(\omega/2)}, \quad \omega \neq 0$$

$$H(e^{j\omega}) = \frac{1}{M} M, \quad \omega = 0$$

Frequency response of LTI system, example

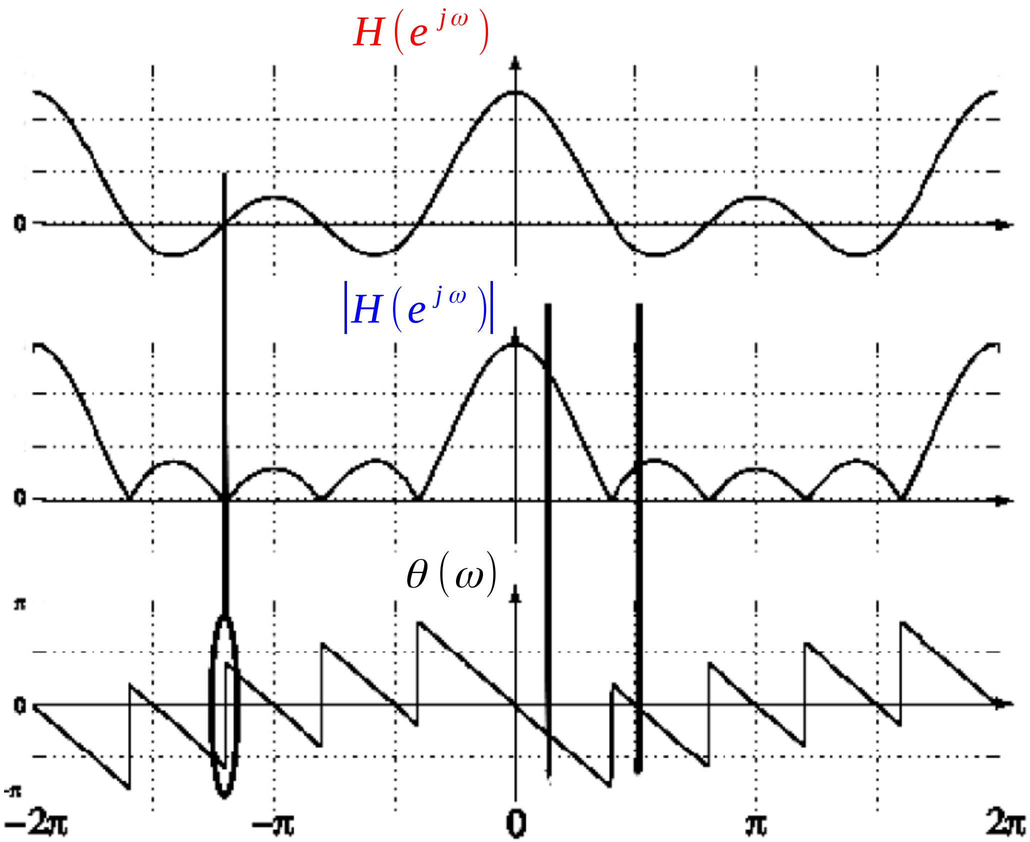
- Frequency response and amplitude response, $M = 5$

$$H(e^{j\omega}) = \frac{1}{M} e^{-j\omega\frac{(M-1)}{2}} \frac{\sin(M\omega/2)}{\sin(\omega/2)}$$

$$H(e^{j\omega}) = \frac{1}{M} M, \quad \omega = 0$$

$$|H(e^{j\omega})| = \left| \frac{1}{M} \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right|, \quad \omega \neq 0$$

$$|H(e^{j\omega})| = \frac{1}{M} M, \quad \omega = 0$$



Frequency response of LTI system, example

- Frequency response and phase response

$$H(e^{j\omega}) = \frac{1}{M} e^{-j\omega\frac{(M-1)}{2}} \frac{\sin(M\omega/2)}{\sin(\omega/2)}$$

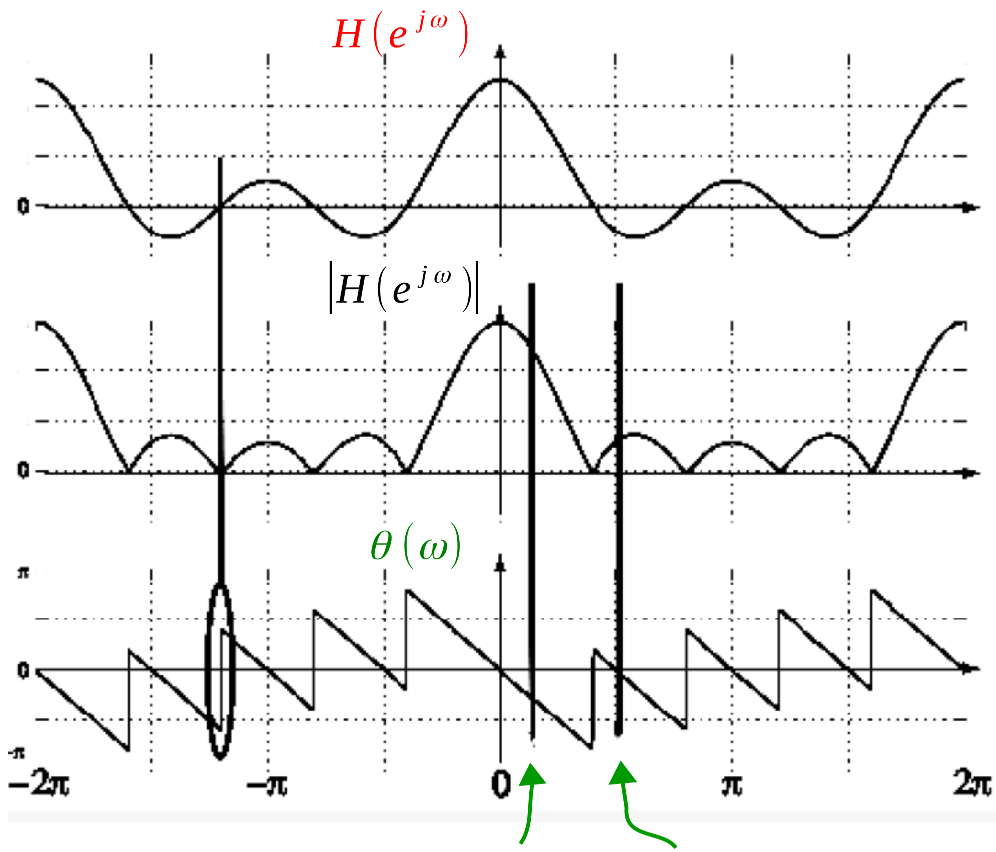
$$\theta(\omega) = \measuredangle H(e^{j\omega})$$

$$\theta(\omega) = \measuredangle \frac{1}{M} + \measuredangle e^{-j\omega\frac{(M-1)}{2}} + \measuredangle \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

$$\theta(\omega) = -\frac{(M-1)}{2}\omega + \pi . r$$

Jumps at sign changes

$$r = \lfloor M\omega / 2\pi \rfloor$$



Response to $\omega_0 = 0.1\,\pi$, $\omega_1 = 0.5\,\pi$ $\longrightarrow$ $x(n) = e^{j\omega_0 n} + e^{j\omega_1 n}$

Frequency response of LTI system, example

- Moving average, $M = 5$
- Input

$$x(n) = e^{j\omega_0 n} + e^{j\omega_1 n}$$

$$\omega_0 = 0.1\pi \rightarrow H(e^{j\omega_0}) \approx 0.8\, e^{j\theta(\omega_0)}$$

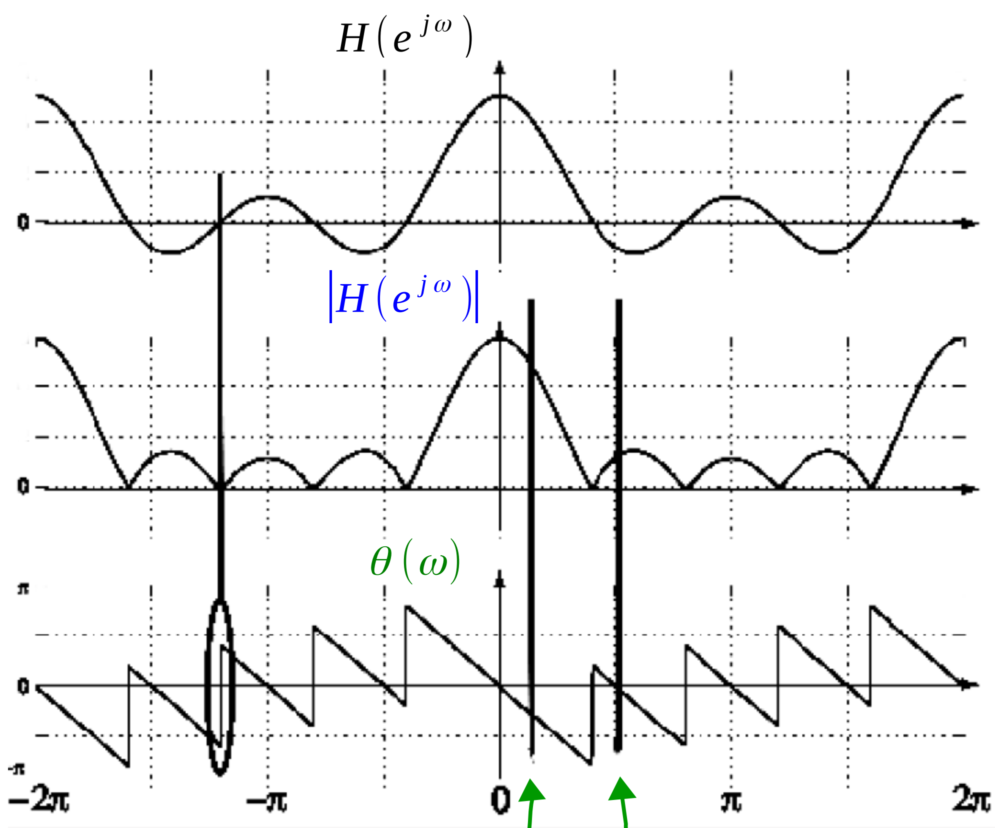
$$\omega_1 = 0.5\pi \rightarrow H(e^{j\omega_1}) \approx (-)\,0.2\, e^{j\theta(\omega_1)}$$

$$\theta(\omega_0) = -2\omega_0 = -2(0.1\pi) = -0.2\pi$$

$$\theta(\omega_1) = -2\omega_1 = -2(0.5\pi) = -\pi$$

- Output

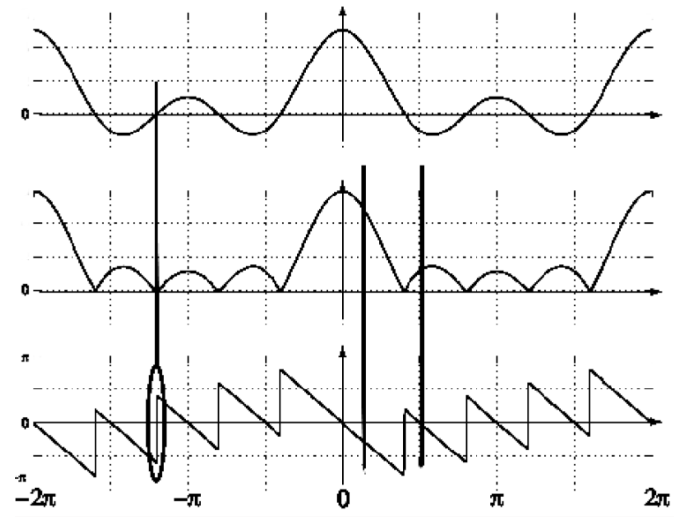
$$y(n) = H(e^{j\omega_0})\, e^{j\omega_0 n} + H(e^{j\omega_1})\, e^{j\omega_1 n}$$



Response to $\omega_0 = 0.1\pi$, $\omega_1 = 0.5\pi$ $\longrightarrow$ $x(n) = e^{j\omega_0 n} + e^{j\omega_1 n}$

Frequency response of LTI system, example

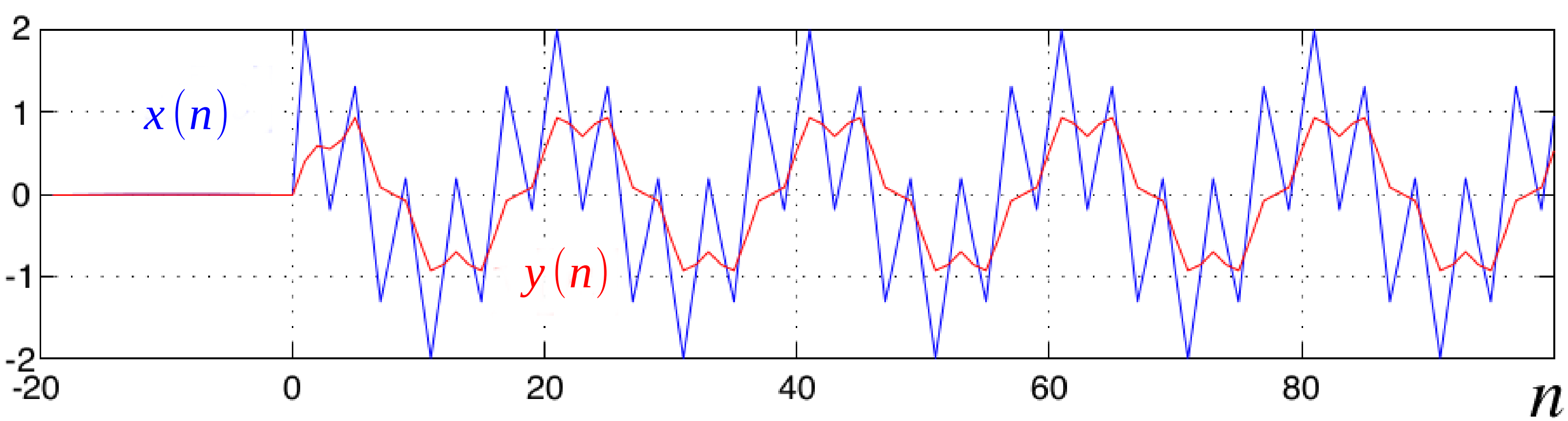
- Moving average, $M = 5$
- Input $$x(n) = e^{j\omega_0 n} + e^{j\omega_1 n}$$



$$\omega_0 = 0.1\pi \rightarrow H(e^{j\omega_0}) \approx 0.8 e^{j\theta(\omega_0)}$$

$$\omega_1 = 0.5\pi \rightarrow H(e^{j\omega_1}) \approx (-)0.2 e^{j\theta(\omega_1)}$$

- Output $$y(n) = H(e^{j\omega_0})\, e^{j\omega_0 n} + H(e^{j\omega_1})\, e^{j\omega_1 n}$$



Frequency response of LTI system

- Example of exam task

  Derive frequency, amplitude and phase response of the following LTI system.

  Also sketch the amplitude and phase response.

$$y(n) = x(n) - x(n-1)$$

  (Hint: Derive the impulse response h(n) of the system first)

- Example of exam task

  Derive frequency, amplitude and phase response of the following LTI system.

  Also sketch the amplitude and phase response.

$$y(n) = x(n) + x(n-1)$$

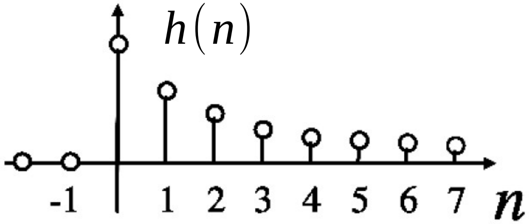
  (Hint: Derive the impulse response h(n) of the system first)

Frequency response of LTI system, example 2, laboratory

- Derive frequency response, amplitude response, and phase response of an LTI system which is defined by the following impulse response

$$h(n) = a^n u(n), \qquad a = 0.6$$

$$y(n) = a\,y(n-1) + x(n)$$



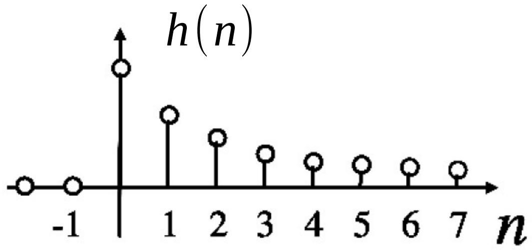
- Draw amplitude and phase response when $a = 0.6$
- Calculate values of the output signal, $y(n)$, amplitude of the output signal, $|H(e^{j\omega})|$, and phase of the output signal, $\theta(\omega)$, in the case of following input signal

$$x(n) = 5 + 12\sin\left(\frac{\pi}{2}n\right) - 20\cos\left(\pi n + \frac{\pi}{4}\right)$$

Frequency response of LTI system, example 2

- Impulse response and frequency response of the system

$$h(n) = a^n u(n), \qquad a = 0.6$$



$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \left(a\, e^{-j\omega}\right)^n = \frac{1}{1 - a\, e^{-j\omega}}$$

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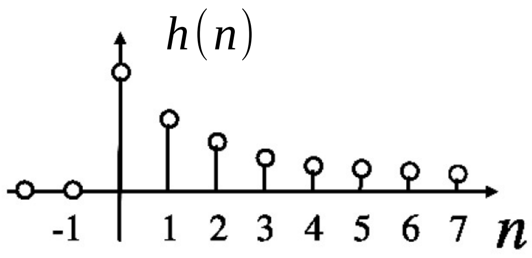
Frequency response of LTI system, example 2

- Frequency, amplitude, and phase response of the system

$$h(n) = a^n u(n), \qquad a = 0.6$$



$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\omega n} = \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}}$$

$$H(e^{j\omega}) = X_R(e^{j\omega}) + X_I(e^{j\omega}) = \frac{1 - a\cos\omega - j a \sin\omega}{1 - 2a\cos\omega + a^2}$$

$$\left|H(e^{j\omega})\right| = \frac{1}{\sqrt{(1 - a\cos\omega)^2 + (a\sin\omega)^2}} = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

$$\theta(\omega) = -\arctan\frac{a\sin\omega}{1 - a\cos\omega}$$

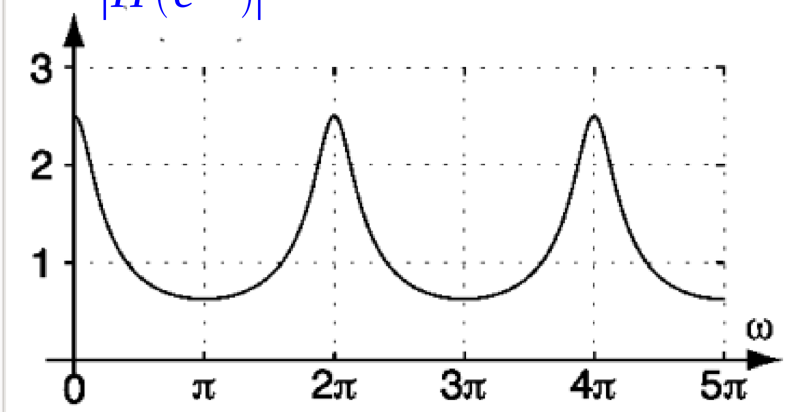
Frequency response of LTI system, example 2

- Draw the amplitude and phase response of the system

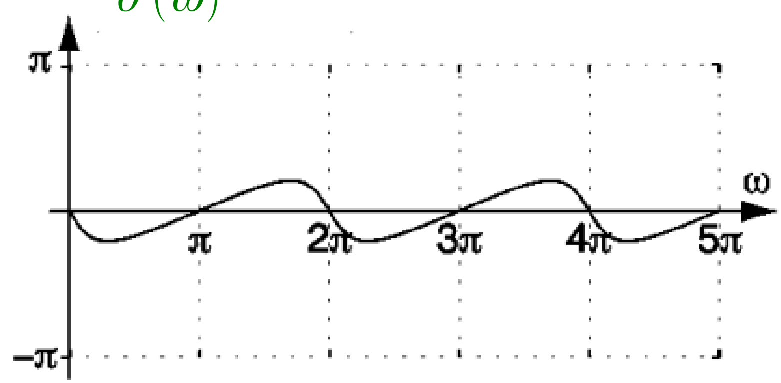
$$h(n) = a^n u(n), \qquad a = 0.6$$

$$\left|H\left(e^{j\omega}\right)\right| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}} \qquad \theta(\omega) = -\arctan\frac{a\sin\omega}{1 - a\cos\omega}$$

$\left|H\left(e^{j\omega}\right)\right|$



$\theta(\omega)$



Frequency response of LTI system, example 2

- Determine the output *y(n)* of the system to the following input signal

$$x(n) = 5 + 12\sin\left(\frac{\pi}{2}n\right) - 20\cos\left(\pi n + \frac{\pi}{4}\right), \qquad \left(\omega = 0,\ \frac{\pi}{2},\ \pi\right)$$

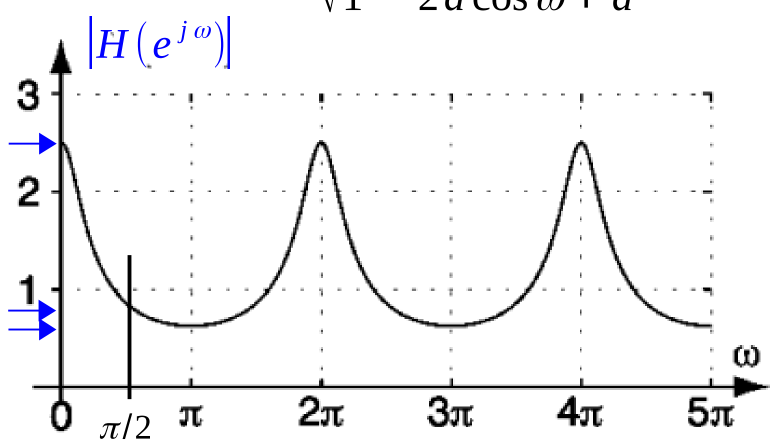
$a = 0.6$

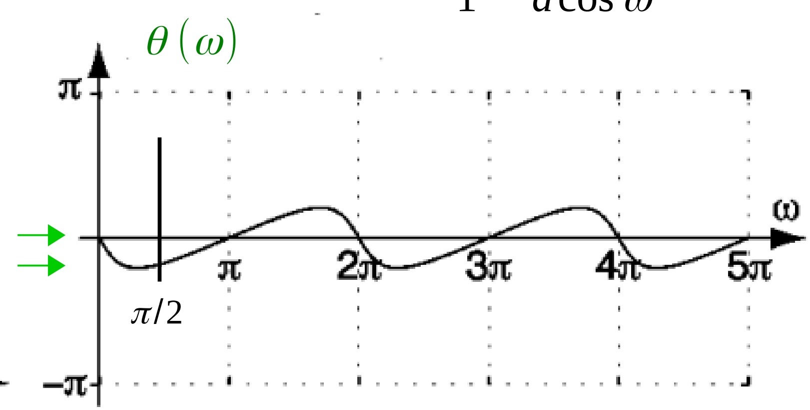
$$\text{For } \omega = 0, \quad |H(0)| = \frac{1}{\sqrt{1 - 2a + a^2}} = 2.5, \qquad \theta(0) = 0^0$$

$$\text{For } \omega = \frac{\pi}{2}, \quad \left|H\left(\frac{\pi}{2}\right)\right| = \frac{1}{\sqrt{1 + a^2}} = 0.86, \qquad \theta\left(\frac{\pi}{2}\right) = -\arctan(a) = -31^0$$

$$\text{For } \omega = \pi, \quad |H(\pi)| = \frac{1}{\sqrt{1 + 2a + a^2}} = 0.63, \qquad \theta(\pi) = 0^0$$

$$|H(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}} \qquad \theta(\omega) = -\arctan\frac{a\sin\omega}{1 - a\cos\omega}$$





Frequency response of LTI system, example 2

- Determine the output *y(n)* of the system to the following input signal

$$x(n) = 5 + 12\sin\left(\frac{\pi}{2}n\right) - 20\cos\left(\pi n + \frac{\pi}{4}\right), \quad \left(\omega = 0, \ \frac{\pi}{2}, \ \pi\right)$$

$a = 0.6$

$$\text{For } \omega = 0, \quad |H(0)| = \frac{1}{\sqrt{1 - 2a + a^2}} = 2.5, \qquad \theta(0) = 0^0$$

$$\text{For } \omega = \frac{\pi}{2}, \quad \left|H\left(\frac{\pi}{2}\right)\right| = \frac{1}{\sqrt{1 + a^2}} = 0.86, \qquad \theta\left(\frac{\pi}{2}\right) = -\arctan(a) = -31^0$$

$$\text{For } \omega = \pi, \quad |H(\pi)| = \frac{1}{\sqrt{1 + 2a + a^2}} = 0.63, \qquad \theta(\pi) = 0^0$$

$$y(n) = 5\,|H(0)| + 12\left|H\left(\frac{\pi}{2}\right)\right|\sin\left[\frac{\pi}{2}n + \theta\left(\frac{\pi}{2}\right)\right] - 20\,|H(\pi)|\cos\left[\pi n + \frac{\pi}{4} + \theta(\pi)\right]$$