Fourier domain, I

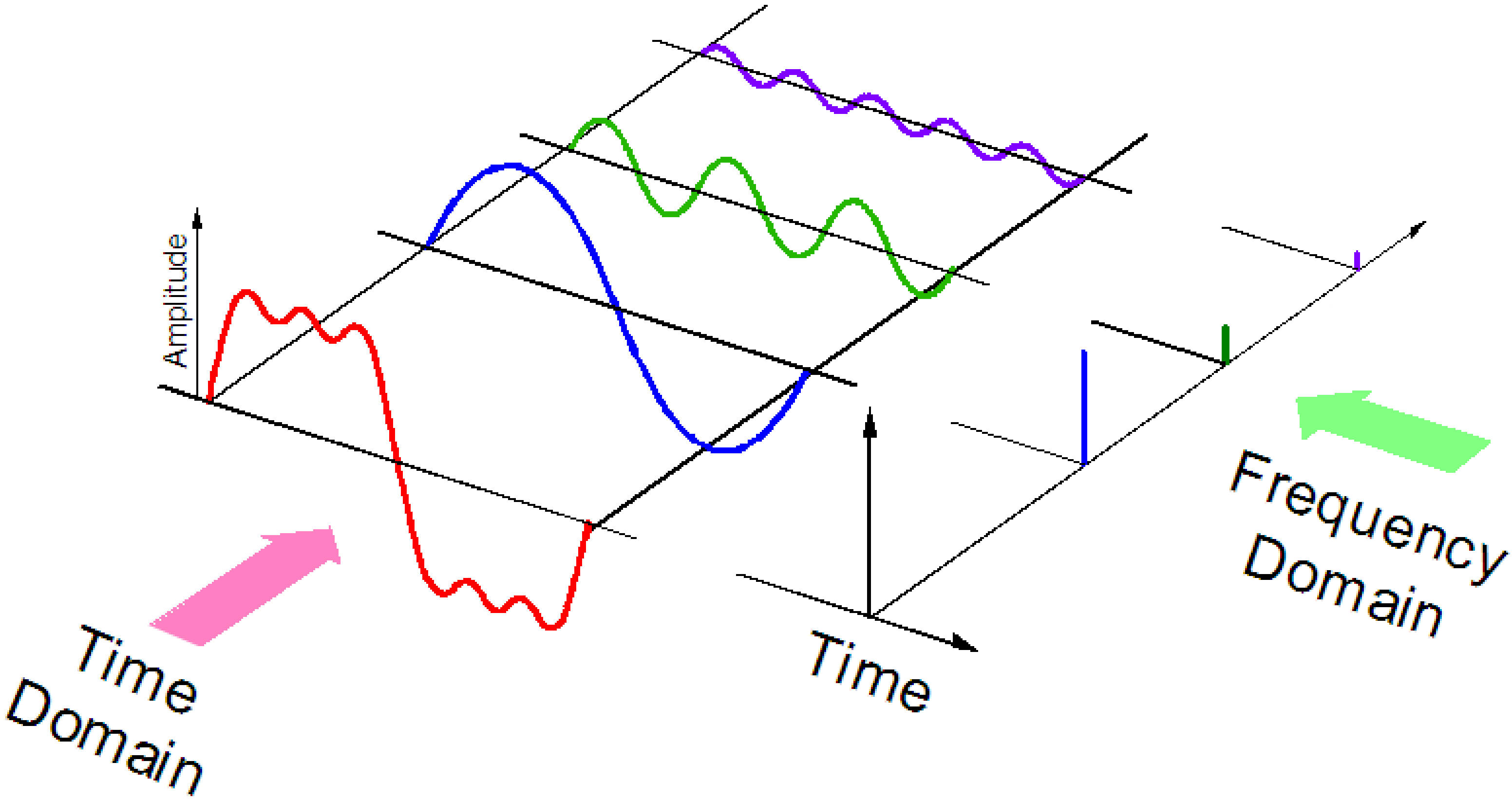
- Introduction
- Frequency domain representation of signals
- Fourier Transforms
- Sinusoids
- Discrete-Time Fourier Transform (DTFT)
- Discrete Fourier Transform (DFT)
- Frequency, amplitude and phase spectrum
- Fourier transform of real signals
- Discrete Fourier Transform (DFT)
- DTFT of rectangular signal
- DFT of rectangular signal
- DFT, zero padding
- Fourier transform properties
- Convolution
- Linear convolution using the DFT
- (Additional materials)

Introduction

- Why Fourier transformation?
  - In many cases frequency-domain representation is more convenient for signal processing, and analysis and synthesis of discrete-time LTI systems
    - * Allows better description of the properties of signals in terms of signal spectrum (decomposition of a signal to frequencies, noise described by frequency components, certain frequency components may contain information related to detection of events)
    - * Allows better description of the properties of LTI systems and their easier design in terms of their frequency response (amplitude characteristic, phase characteristic, physical meaning of frequency response)
- Frequency domain signal processing, analysis and synthesis
  - Signal spectrum of a signal → Useful information about the signal
  - Frequency response of an LTI system → Determines change of frequency content of a signal while passing the system

Frequency-domain representation of signals

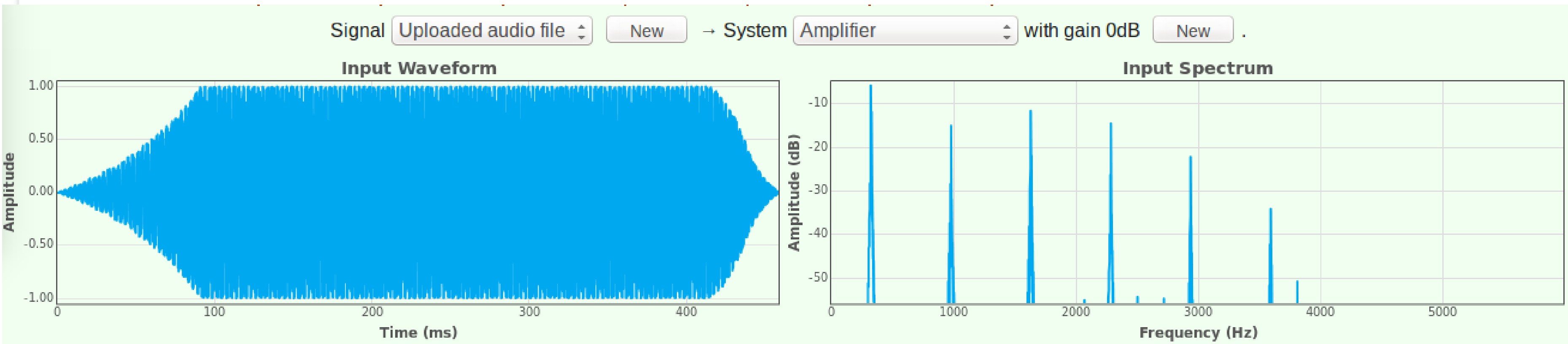
- Fourier (1807) "… all periodic continuous signals can be represented by a sum of properly chosen sinus signals …"

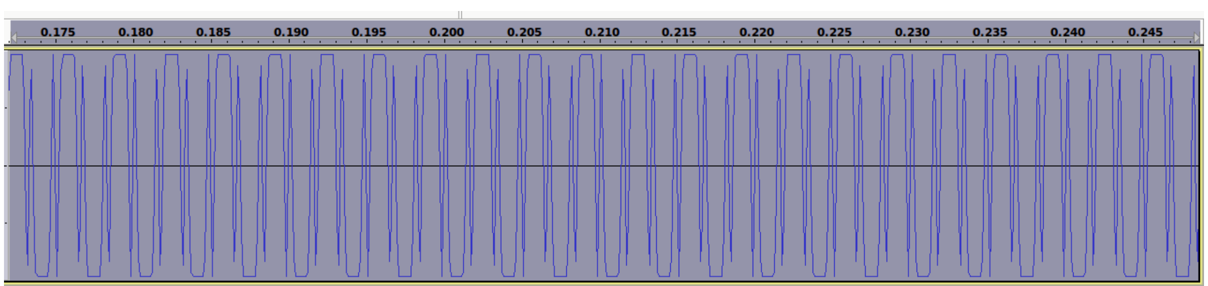


Frequency-domain representation of signals

- Examples of frequency representation of signals (amplitude spectrum)

Clarinet (clarinet.wav)





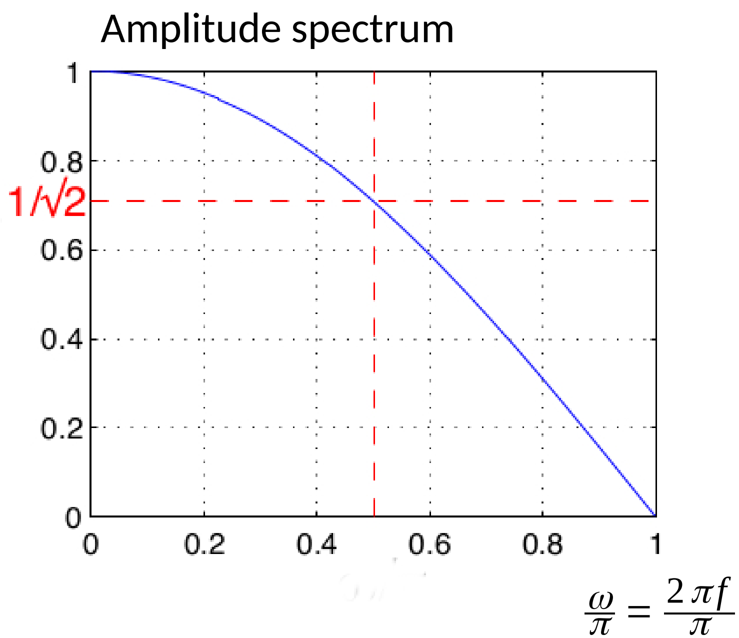
clarinet.wav
Play

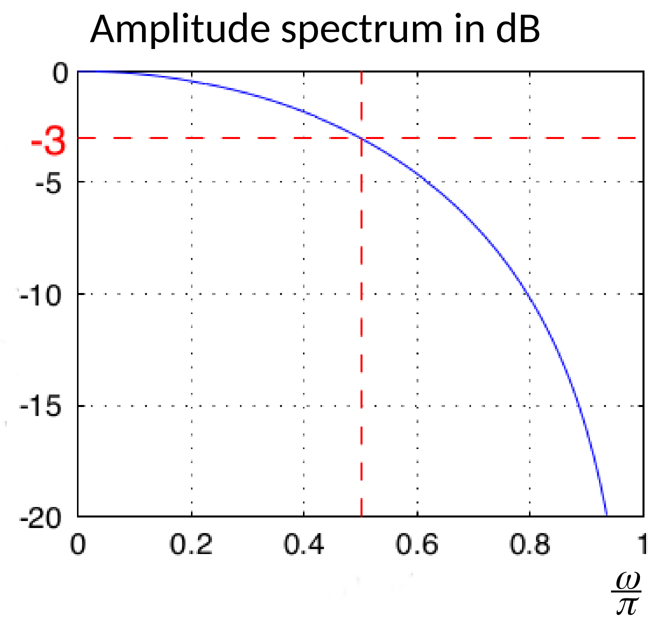
- http://www.speechandhearing.net/laboratory/esystem/

Frequency-domain representation of signals

- DeciBels
- Level, *level*, of amplitude spectrum is often expressed in deciBels (dB)
- dB is simply a scaled log value (level of 0 corresponds to $-\infty$ dB)

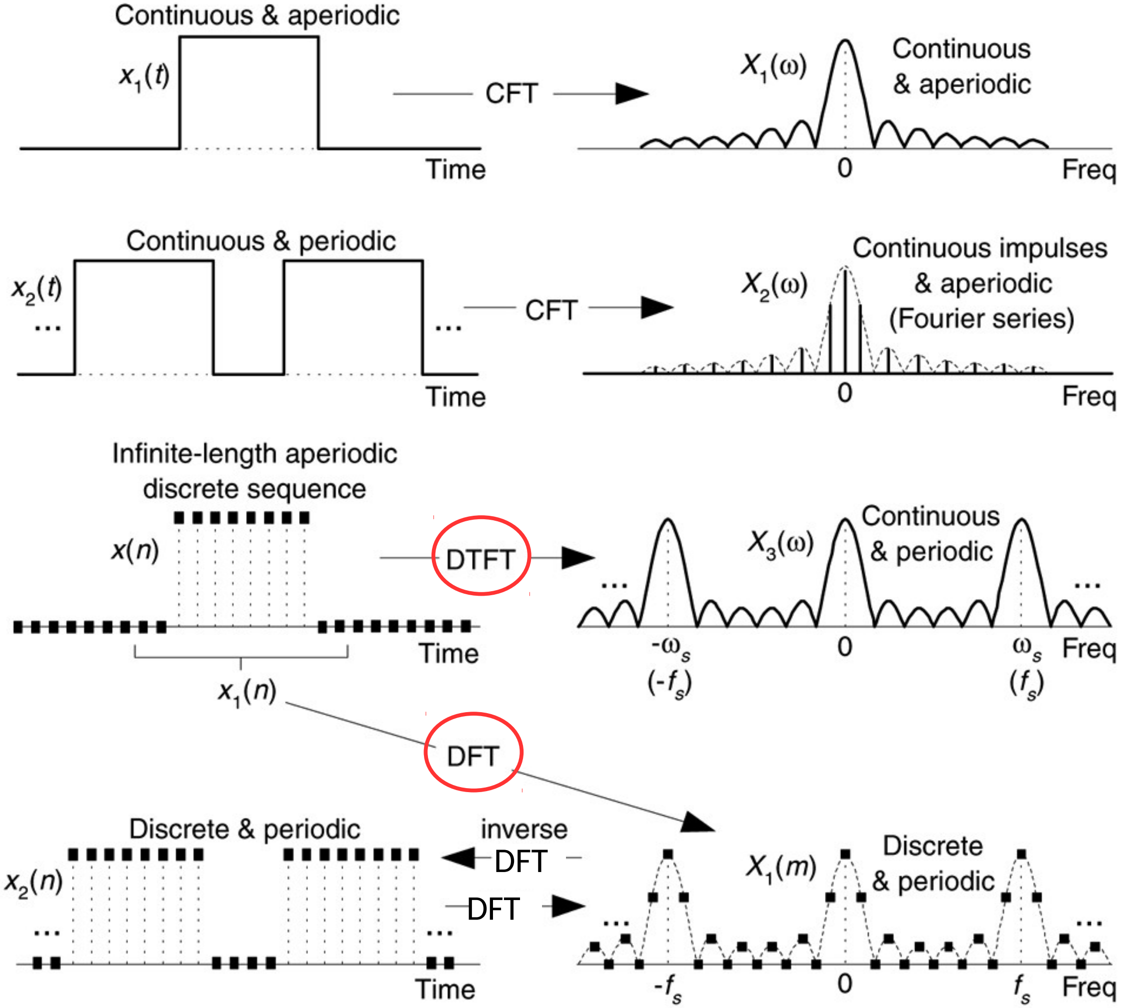
$$dB = 20.\log_{10}(level) = 10.\log_{10}(level^2) = 10.\log_{10}(power)$$





Fourier transforms

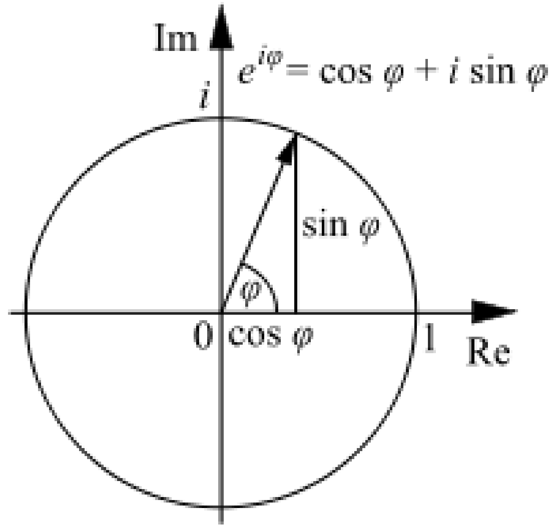
1.
2.
3.
4.

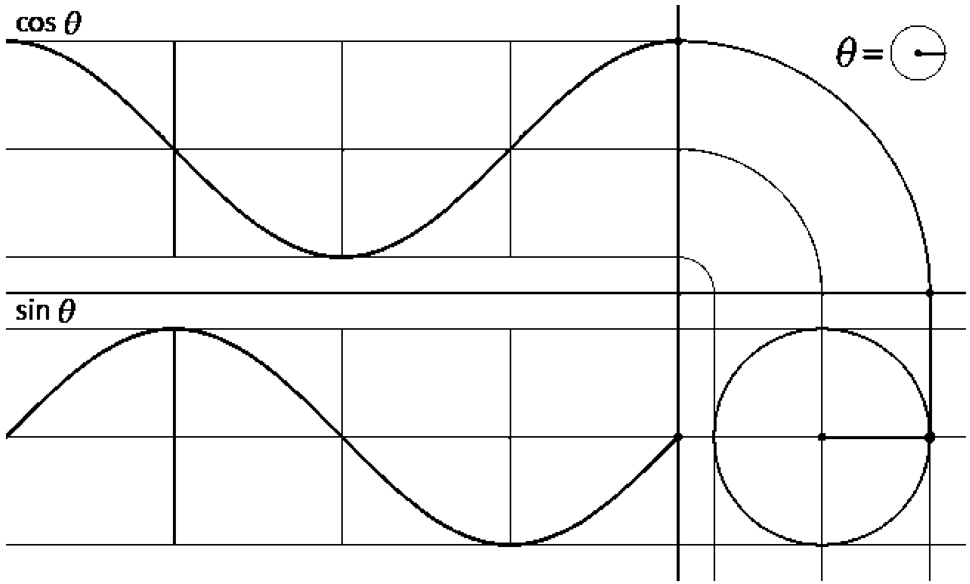


Sinusoids

- Sinusoids and complex exponential functions are joined by Euler equation

$$e^{\pm jx} = \cos(x) \pm j\sin(x)$$





$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$j = i = \sqrt{-1}$$

Complex sinusoids - review https://www.youtube.com/watch?v=GhhRIjMywu0

Discrete-Time Fourier Transform (DTFT)

- How to find frequency spectrum $X(e^{j\omega})$ of a discrete signal *x(n)*?

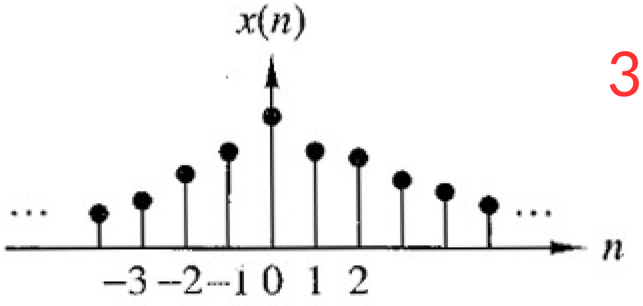
  Complex sinusoids $e^{-j\omega}$ pick out (probe) the corresponding sinusoids that are linearly combined in *x(n)*

$$e^{-j\omega} = \cos(\omega) - j\sin(\omega)$$

- The DTFT of a discrete signal *x(n)*

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad -\pi < \omega < \pi, \;\; \omega = 2\pi f, \;\; -1/2 < f < 1/2$$

summation, argument is $e^{j\omega}$ → continuous and periodic with a period of ω (2π), f (1)



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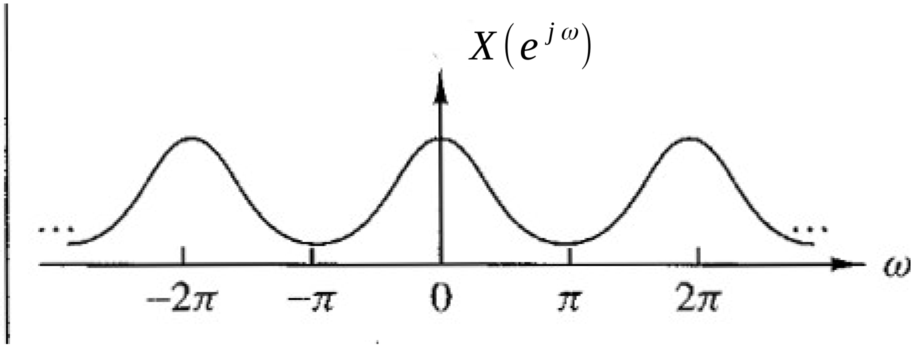
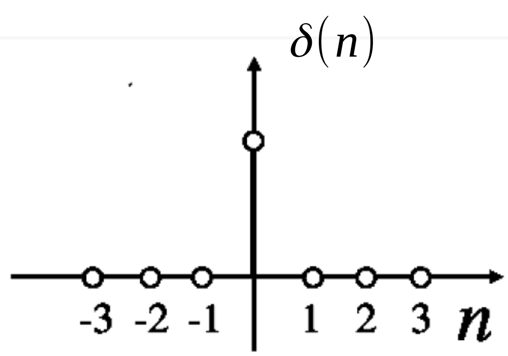
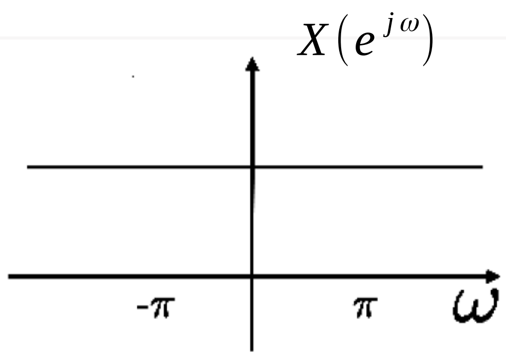


Discrete-Time Fourier Transform (DTFT)

- Example, discrete-time Fourier transforms of the unit sample, $x(n) = \delta(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta(n).e^{-j\omega n} = \delta(0).e^{-j\omega 0} = 1$$





Discrete Fourier Transform (DFT)

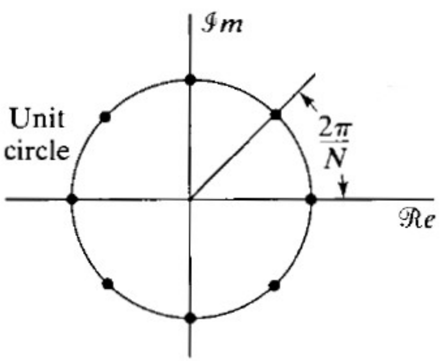
- A finite (or periodic) sequence has only *N* unique values, *x(n)*, 0 ≤ *n* ≤ *N*

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$$

- Spectrum is defined by *N* distinct frequency samples $\omega_k = 2\pi k / N$
- The DFT divides (or samples) interval 0 .. 2*π* into *N* equal steps to get *X(k)*

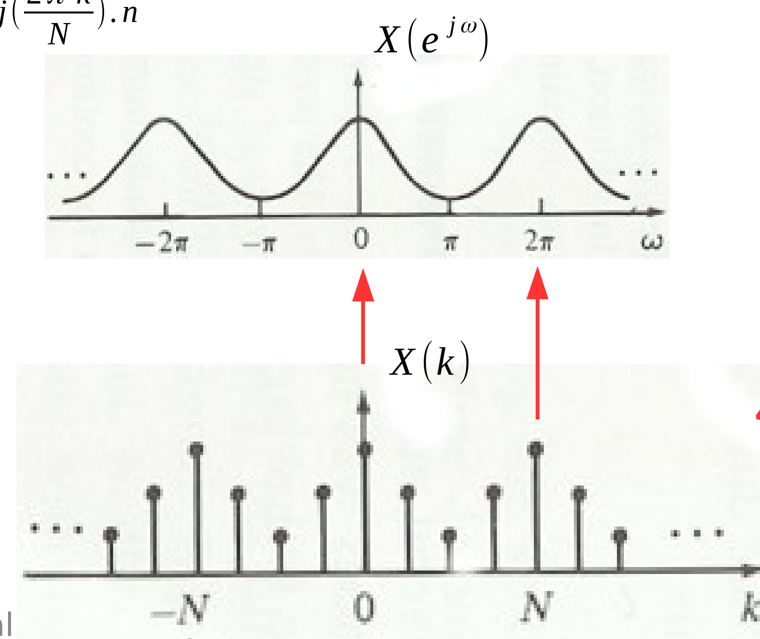
$$X(k) \equiv X(e^{j\omega})\Big|_{\omega_k = \frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi k}{N}\right).n}$$

$$k = 0,1,2,\ldots, N-1$$



te, discrete
periodic with
eriod of *N*

Example, *N* = 4 →



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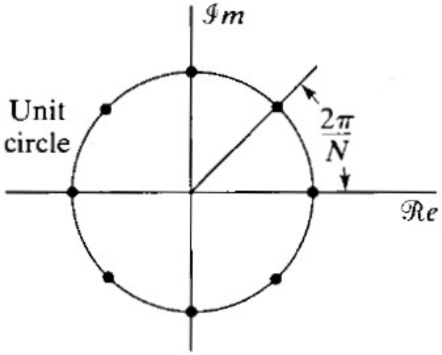
Discrete Fourier Transform (DFT)

- The DFT samples $X(e^{j\omega})$ at points on the unit circle to get *X(k)*

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j(\frac{2\pi k}{N}).n} \qquad \omega_k = \frac{2\pi k}{N}$$



- *X(k)* – *N* samples of frequency spectrum, *k* = 0,1,2, ..., *N-1*
- *ΔF* = *Fs* / N - Interval between two samples of frequency spectrum [*Hz*]
- F_k = *k* . *ΔF* - Frequency of *k*-th sample of frequency spectrum [*Hz*]

- Inverse DFT (IDFT)

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j(\frac{2\pi n}{N}).k}$$

→ finite, discrete and periodic with a period of N

Discrete Fourier Transform (DFT)

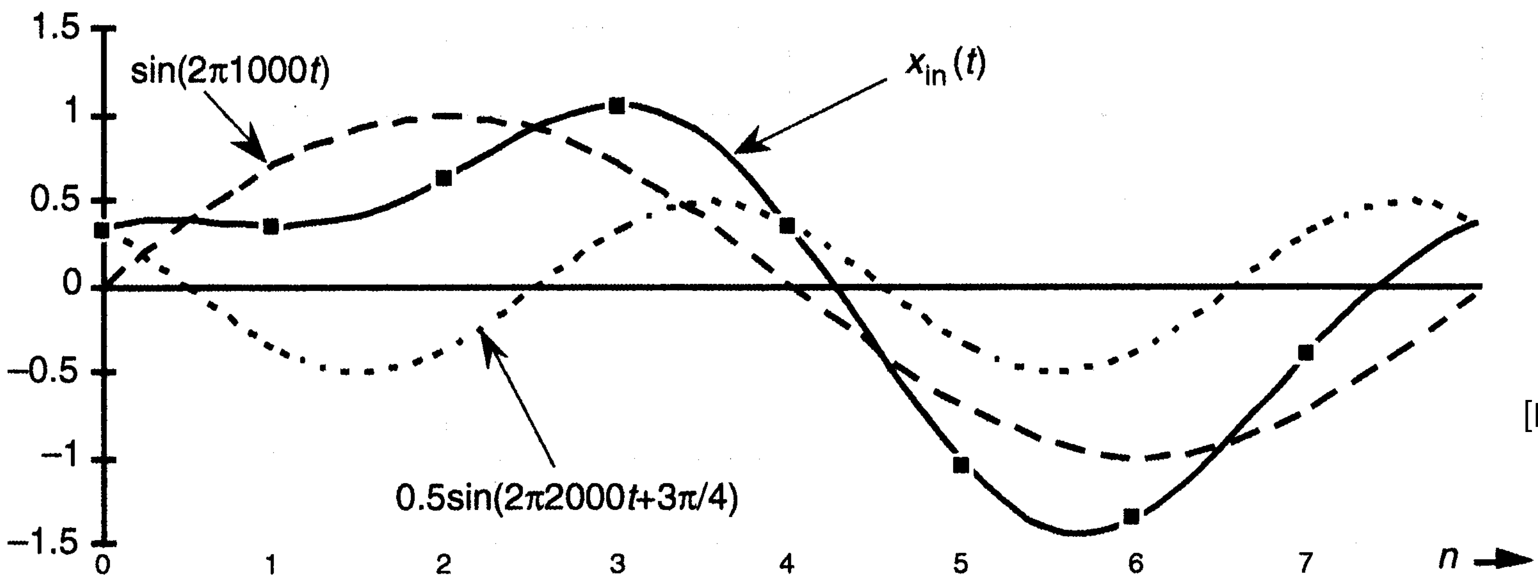
- DFT example

$$x(n) = x_{in}(nT_S) = \sin(2\pi.1000.nT_S) + 0.5\sin(2\pi.2000.nT_S + 3\pi/4)$$

Sampling frequency, $F_S = 1 / T_S = 8000$ smp/sec

$$x(0) = 0.3535, \quad x(1) = 0.3535, \quad x(2) = 0.6464, \quad x(3) = 1.0607$$

$$x(4) = 0.3535, \quad x(5) = -1.0607, \quad x(6) = -1.3535, \quad x(7) = -0.3535$$



[Lyons]

Discrete Fourier Transform (DFT)

- DFT example

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k}{N}.n}$$

$$X(k) = \sum_{n=0}^{N-1} (x(n)\cos(\frac{2\pi k}{N}n) - j\,x(n)\sin(\frac{2\pi k}{N}n))$$

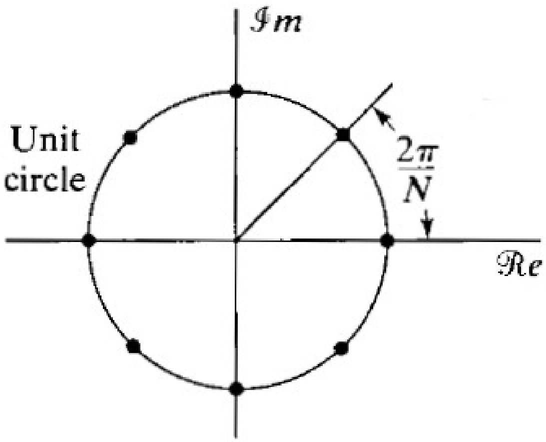
$$X(0) = \sum_{n=0}^{7} x(n)$$

$$X(1) = \sum_{n=0}^{7} (x(n)\cos(\frac{2\pi 1}{8}n) - j\,x(n)\sin(\frac{2\pi 1}{8}n))$$

$$X(2) = \sum_{n=0}^{7} (x(n)\cos(\frac{2\pi 2}{8}n) - j\,x(n)\sin(\frac{2\pi 2}{8}n))$$

...

$$X(7) = \sum_{n=0}^{7} (x(n)\cos(\frac{2\pi 7}{8}n) - j\,x(n)\sin(\frac{2\pi 7}{8}n))$$



DFT

- DFT example

$$X(0) = \sum_{n=0}^{7} x(n)$$

$$X(0) = 0.0 = 0 \measuredangle 0^0$$

$$X(1) = \sum_{n=0}^{7} \left(x(n)\cos\left(\frac{2\pi 1}{8}n\right) - j\,x(n)\sin\left(\frac{2\pi 1}{8}n\right)\right)$$

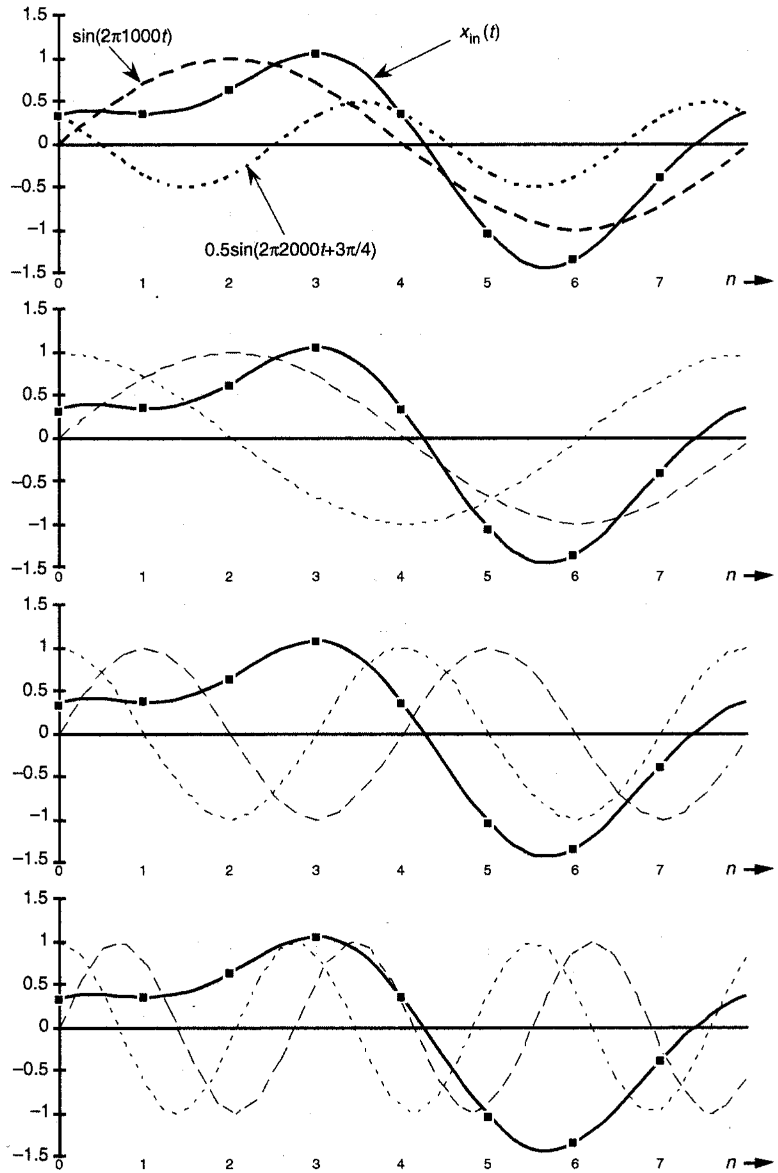
$$X(1) = 0.0 - j4.0 = 4 \measuredangle -90^0$$

$$X(2) = \sum_{n=0}^{7} \left(x(n)\cos\left(\frac{2\pi 2}{8}n\right) - j\,x(n)\sin\left(\frac{2\pi 2}{8}n\right)\right)$$

$$X(2) = 1.414 + j1.414 = 2 \measuredangle 45^0$$

$$X(3) = \sum_{n=0}^{7} \left(x(n)\cos\left(\frac{2\pi 3}{8}n\right) - j\,x(n)\sin\left(\frac{2\pi 3}{8}n\right)\right)$$

$$X(3) = 0.0 - j0.0 = 0 \measuredangle 0^0$$



DFT

- DFT example

$$X(4) = \sum_{n=0}^{7} \left(x(n) \cos\left(\frac{2\pi 4}{8} n\right) - j\, x(n) \sin\left(\frac{2\pi 4}{8} n\right)\right)$$

$$X(4) = 0.0 - j\,0.0 = 0 \measuredangle 0^0$$

$$X(5) = \sum_{n=0}^{7} \left(x(n) \cos\left(\frac{2\pi 5}{8} n\right) - j\, x(n) \sin\left(\frac{2\pi 5}{8} n\right)\right)$$

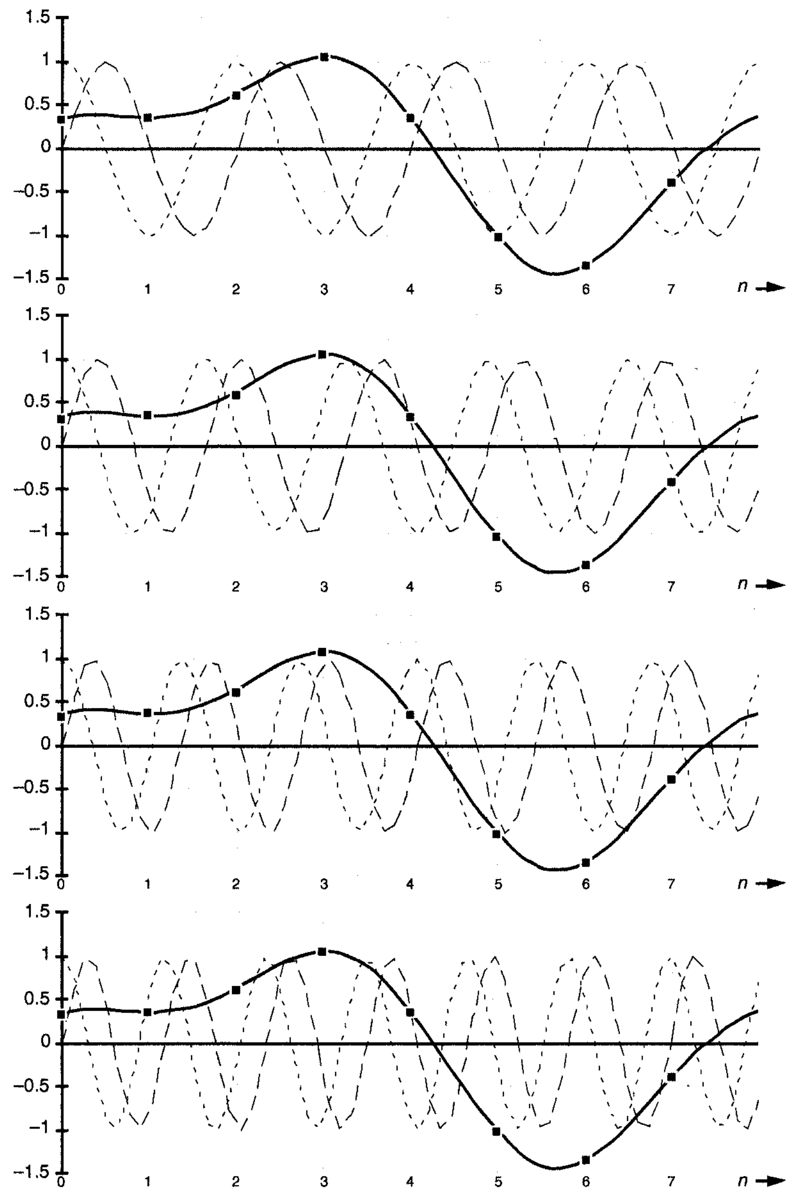
$$X(5) = 0.0 - j\,0.0 = 0 \measuredangle 0^0$$

$$X(6) = \sum_{n=0}^{7} \left(x(n) \cos\left(\frac{2\pi 6}{8} n\right) - j\, x(n) \sin\left(\frac{2\pi 6}{8} n\right)\right)$$

$$X(6) = 1.414 - j\,1.414 = 2 \measuredangle -45^0$$

$$X(7) = \sum_{n=0}^{7} \left(x(n) \cos\left(\frac{2\pi 7}{8} n\right) - j\, x(n) \sin\left(\frac{2\pi 7}{8} n\right)\right)$$

$$X(7) = 0.0 + j\,4.0 = 4 \measuredangle 90^0$$



Discrete Fourier Transform (DFT)

- DFT example

$$X(0) = 0.0 - j0.0 = 0 \angle 0^0$$

$$X(1) = 0.0 - j4.0 = 4 \angle -90^0$$

$$X(2) = 1.414 + j1.414 = 2 \angle 45^0$$

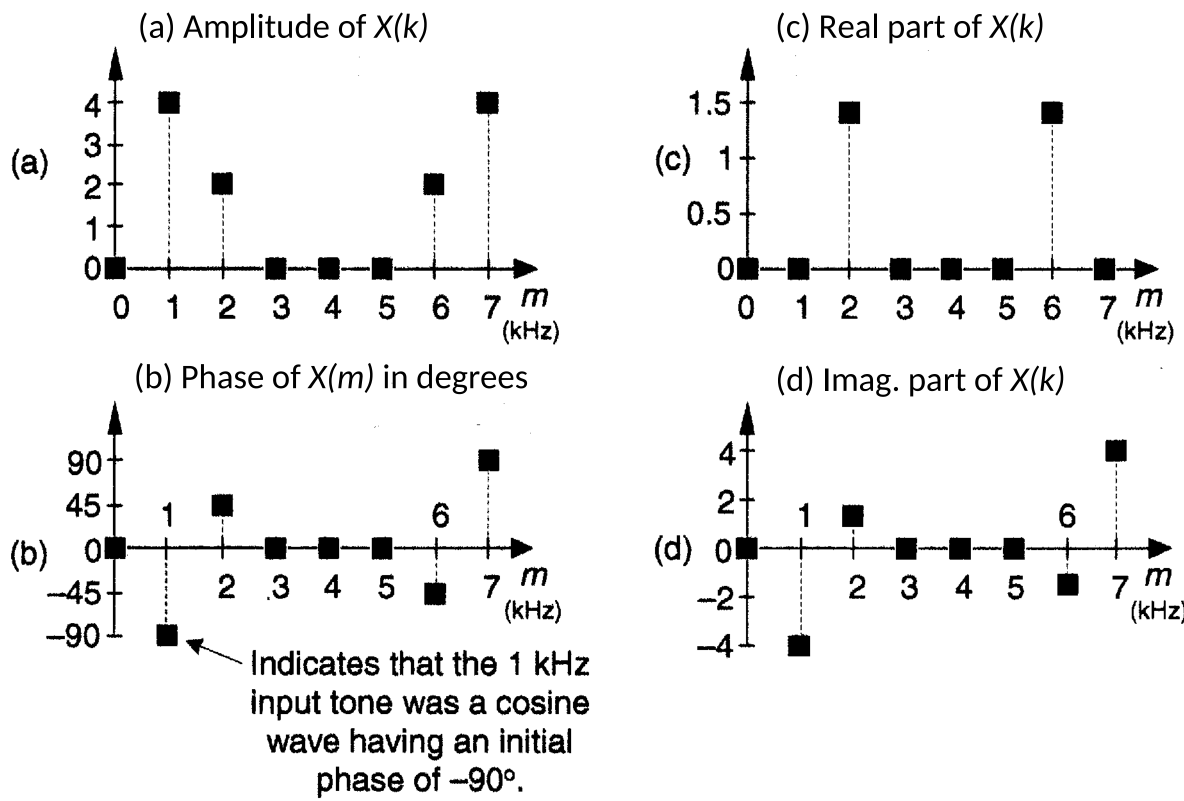
$$X(3) = 0.0 - j0.0 = 0 \angle 0^0$$

$$X(4) = 0.0 - j0.0 = 0 \angle 0^0$$

$$X(5) = 0.0 - j0.0 = 0 \angle 0^0$$

$$X(6) = 1.414 - j1.414 = 2 \angle -45^0$$

$$X(7) = 0.0 + j4.0 = 4 \angle 90^0$$



For real signals: $X(m) = X^*(N-m), \quad m = 1,2,\ldots,N-1$

Frequency, amplitude and phase spectrum

- The Fourier transform of a discrete signal, *x(n)*, is called the frequency spectrum, it is a sequence of complex numbers

DTFT | DFT

- Frequency spectrum

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) \qquad X(k) = X_R(k) + jX_I(k)$$

$$X(e^{j\omega}) = |X(e^{j\omega})| \,.\, e^{j\theta(\omega)} \qquad X(k) = |X(k)| \,.\, e^{j\theta(k)}$$

- Amplitude spectrum:

$$|X(e^{j\omega})| = \sqrt{X_R^2(e^{j\omega}) + X_I^2(e^{j\omega})} \qquad |X(k)| = \sqrt{X_R^2(k) + X_I^2(k)}$$

- Phase spectrum:

$$\theta(\omega) = \sphericalangle X(e^{j\omega}) = \arctan\left(\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right) \qquad \theta(k) = \sphericalangle X(k) = \arctan\left(\frac{X_I(k)}{X_R(k)}\right)$$

- Power spectrum:

$$|X(e^{j\omega})|^2 = X_R^2(e^{j\omega}) + X_I^2(e^{j\omega}) \qquad |X(k)|^2 = X_R^2(k) + X_I^2(k)$$

Fourier transform of real signals

- Fourier transform pair (DFT)

$$x(n) \leftrightarrow X(k)$$

- If *x(n)* is real signal

$$X(k) = X^*(N-k) \quad \text{Conjugate symmetric}$$

$$X_R(k) = X_R(N-k) \quad \text{Real part is even}$$

$$X_I(k) = -X_I(N-k) \quad \text{Imaginary part is odd}$$

$$|X(k)| = |X(N-k)| \quad \text{Amplitude spectrum is even}$$

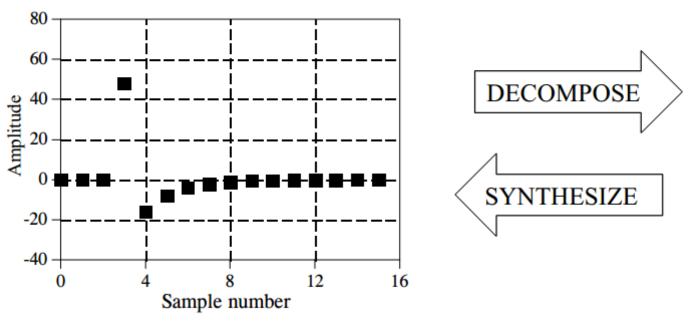
$$\measuredangle X(k) = -\measuredangle X(N-k) \quad \text{Phase spectrum is odd}$$

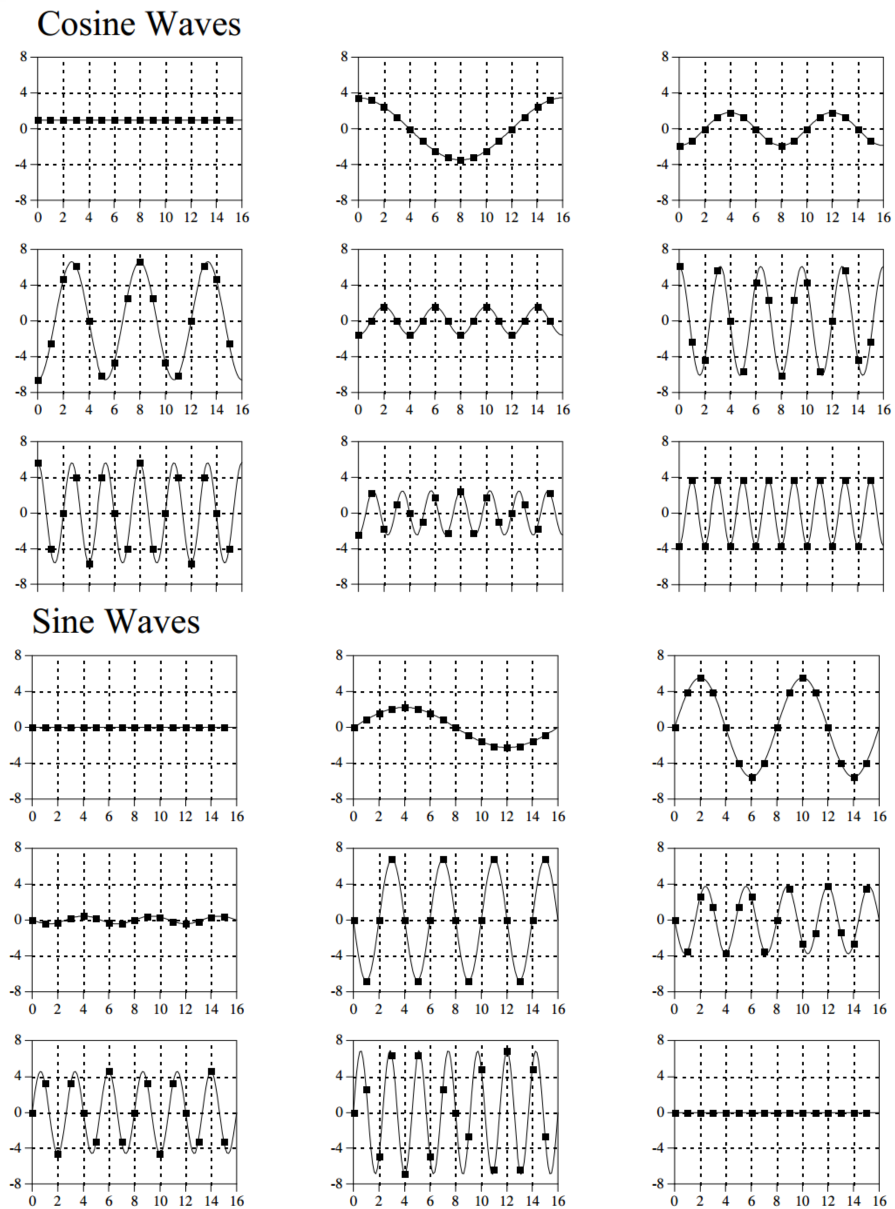
Discrete Fourier Transform (DFT)

- Verify basis functions of the DFT
  - DFT (decomposition)
  - IDFT (synthesis)

  for a discrete signal at

- http://www.fourier-series.com/





Discrete Fourier Transform (DFT)

- Example of exam task

Calculate the four-point Discrete Fourier Transform (DFT) for the following signal, $x(n) = \{\mathbf{1}, 1, 0, 0\}$. Calculate also the amplitude and phase spectrum.

DTFT of rectangular signal

- Rectangular signal

$$x(n) = A.u(n) - A.u(n-L), \quad A=1, \quad L=5$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{L-1} A.e^{-j\omega n}$$

$$X(e^{j\omega}) = A\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}$$

$$X(e^{j\omega}) = AL, \quad \omega = 0$$

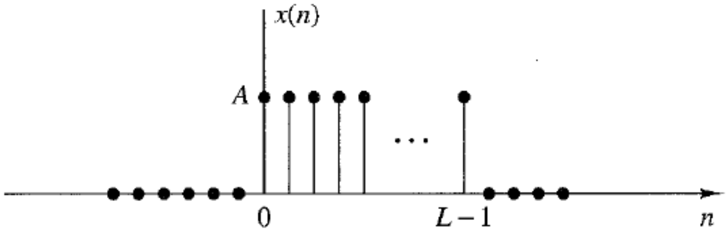
$$X(e^{j\omega}) = A\, e^{-j\omega(L-1)/2}\frac{\sin(\omega L/2)}{\sin(\omega/2)}, \ \omega \neq 0$$

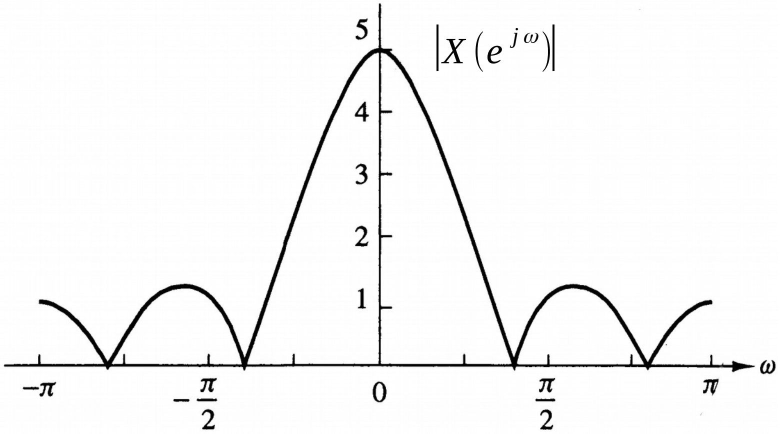
$$|X(e^{j\omega})| = |A|L, \quad \omega = 0$$

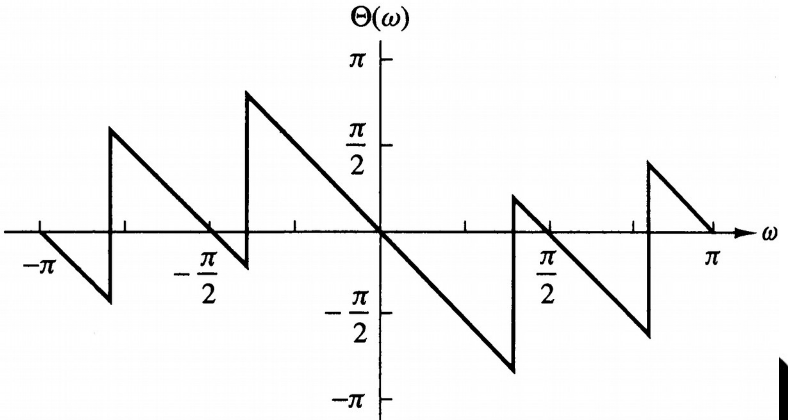
$$|X(e^{j\omega})| = |A|\left|\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right|, \ \omega \neq 0$$

$$\theta(\omega) = \sphericalangle X(e^{j\omega}) = \frac{-(L-1)}{2}\omega + \pi r$$

Jumps at sign changes $\quad r = \lfloor L\omega/2\pi \rfloor$







DTFT of rectangular signal

- Rectangular signal

$$x(n) = A.u(n) - A.u(n-L), \quad A = 1, \; L = 10$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{L-1} A.e^{-j\omega n}$$

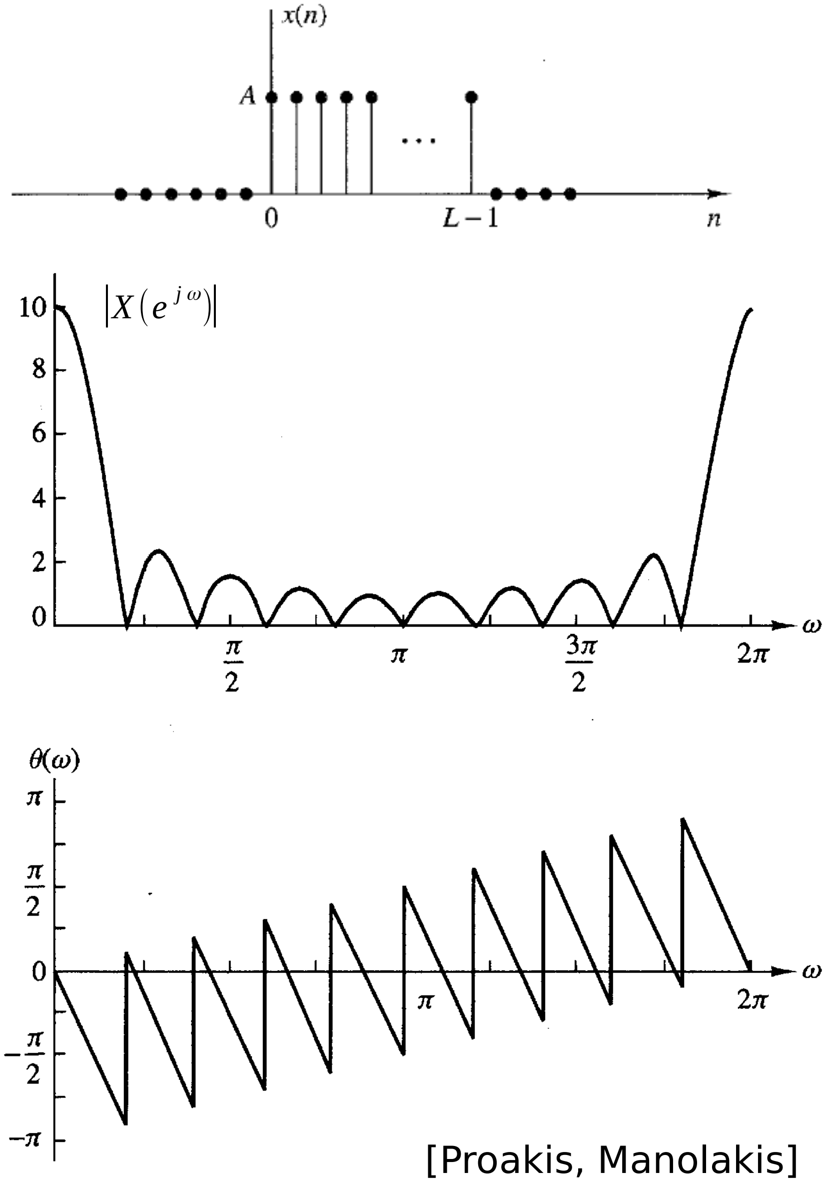
$$\left|X(e^{j\omega})\right| = |A| L, \quad \omega = 0$$

$$\left|X(e^{j\omega})\right| = |A| \left| \frac{\sin(\omega L/2)}{\sin(\omega/2)} \right|, \; \omega \neq 0$$

$$\theta(\omega) = \measuredangle X(e^{j\omega}) = \frac{-(L-1)}{2}\, \omega + \pi r$$

Jumps at sign changes

$$r = \lfloor L\omega / 2\pi \rfloor$$



[Proakis, Manolakis]

DFT of rectangular signal

- Rectangular signal

$$x(n) = A.u(n) - A.u(n-L), \quad A = 1, \; L = 10$$

$$X(k) = \sum_{n=0}^{N-1} A.e^{-j(\omega_k)n}, \quad \omega_k = \frac{2\pi k}{N}$$

$$N = 50, \quad N > L \; !$$

$$X(k) = AL, \quad \omega_k = 0$$

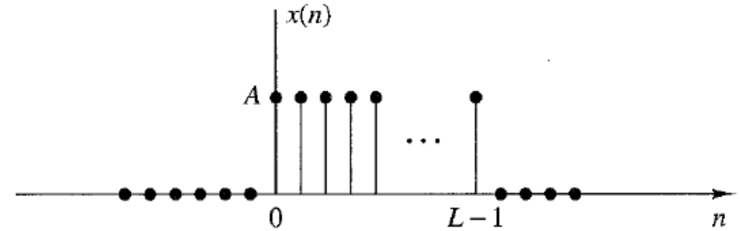
$$X(k) = A\, e^{-j(\omega_k)(L-1)/2} \frac{\sin(\omega_k L/2)}{\sin(\omega_k /2)}, \quad \omega_k \neq 0$$

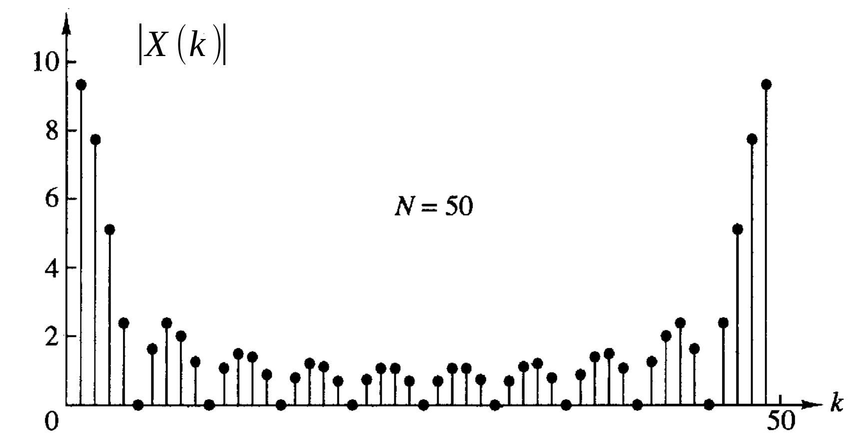
$$|X(k)| = |A| L, \quad \omega_k = 0$$

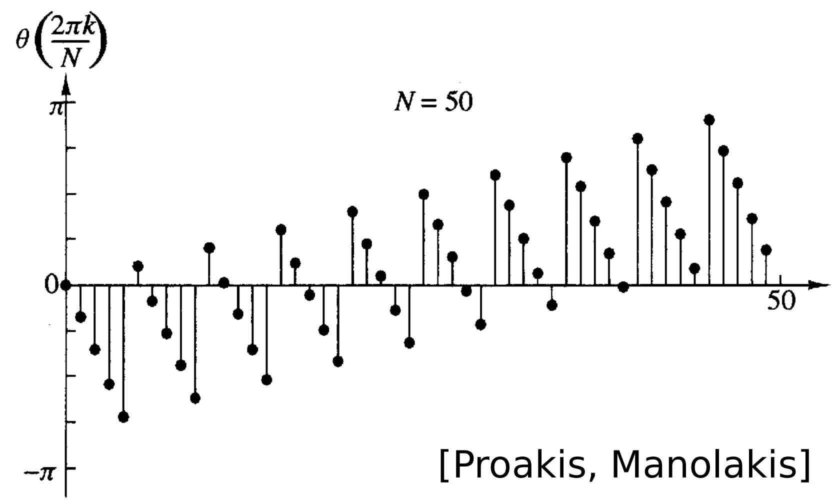
$$|X(k)| = |A| \left| \frac{\sin(\omega_k L/2)}{\sin(\omega_k /2)} \right|, \quad \omega_k \neq 0$$

$$\theta(\omega_k) = \measuredangle X(k) = \frac{-(L-1)}{2} \omega_k + \pi r$$

$$r = \lfloor L \omega_k / 2\pi \rfloor$$







[Proakis, Manolakis]

DFT, zero padding

- Rectangular signal

$$x(n) = A.u(n) - A.u(n-L), \quad A = 1, \; L = 10$$

$$X(k) = \sum_{n=0}^{N-1} A.e^{-j(\omega_k)n}, \quad \omega_k = \frac{2\pi k}{N}$$

$$N = 100, \quad N > L \; !$$

$$X(k) = AL, \quad \omega_k = 0$$

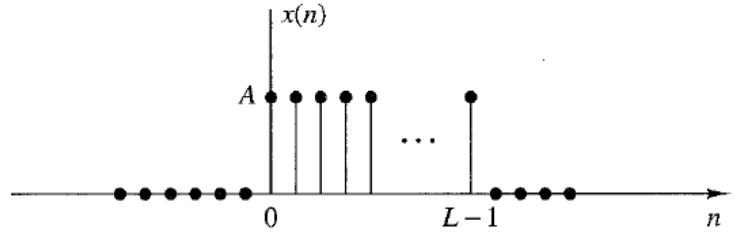
$$X(k) = A\, e^{-j(\omega_k)(L-1)/2} \frac{\sin(\omega_k L/2)}{\sin(\omega_k/2)}, \quad \omega_k \neq 0$$

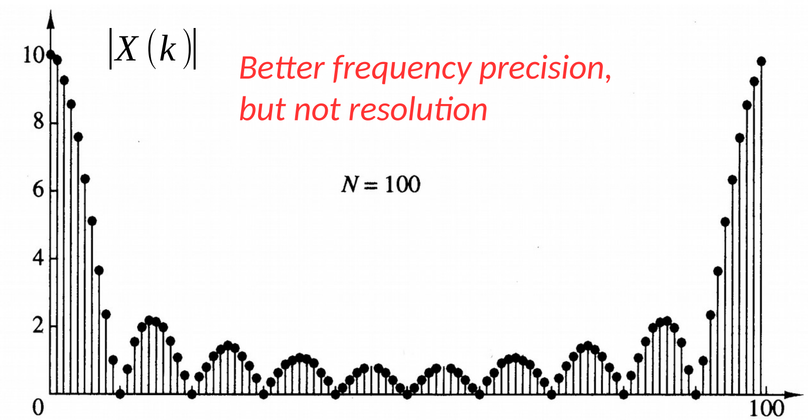
$$|X(k)| = |A| L, \quad \omega_k = 0$$

$$|X(k)| = |A| \left| \frac{\sin(\omega_k L/2)}{\sin(\omega_k/2)} \right|, \quad \omega_k \neq 0$$

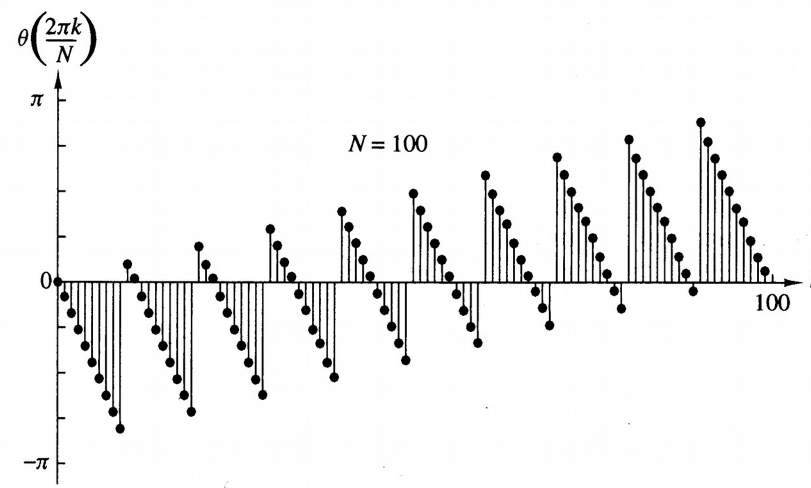
$$\theta(\omega_k) = \measuredangle X(k) = \frac{-(L-1)}{2}\omega_k + \pi r$$

$$r = \lfloor L\omega_k / 2\pi \rfloor$$





Better frequency precision, but not resolution



Fourier transform properties

- Fourier transform pair

$$x(n) \leftrightarrow X(e^{j\omega})$$

Linearity $$a_1 x_1(n) + a_2 x_2(n) \leftrightarrow a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

Time shift $$x(n-n_0) \leftrightarrow X(e^{j\omega}) e^{-j\omega n_0}$$

Time convolution $$x_1(n) * x_2(n) \leftrightarrow X_1(e^{j\omega}) \, . \, X_2(e^{j\omega})$$

Convolution

- Convolution in time domain becomes multiplication in frequency domain

- Since $$y(n) = x(n) * h(n) \leftrightarrow X(e^{j\omega}) \,.\, H(e^{j\omega}) = Y(e^{j\omega})$$

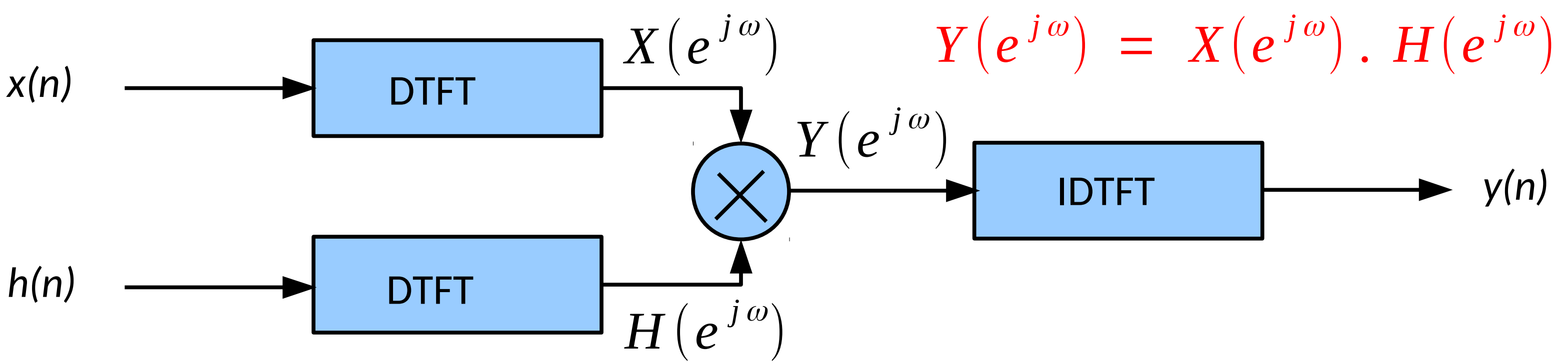
convolution can be calculated following:

(1) find DTFT of *x(n)* and *h(n)* → $X(e^{j\omega})$ and $H(e^{j\omega})$

(2) multiply $X(e^{j\omega}) \,.\, H(e^{j\omega})$

(3) derive IDTFT of the product → *y(n)* = *x(n)* * *h(n)*

$$Y(e^{j\omega}) = X(e^{j\omega}) \,.\, H(e^{j\omega})$$

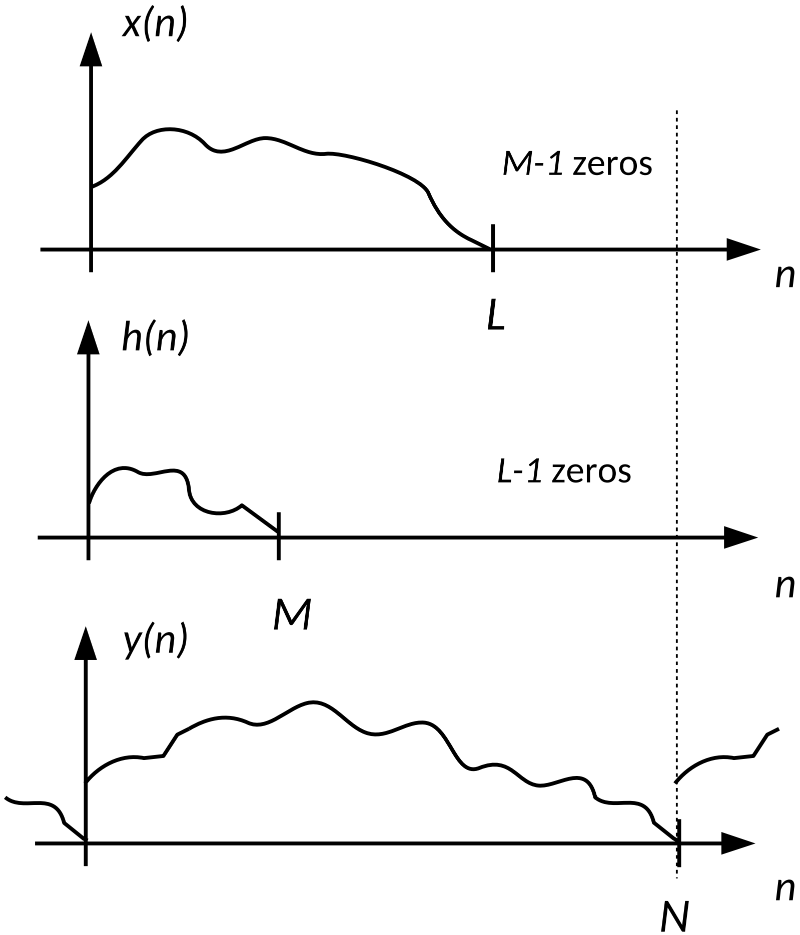


The same with DFT (*X(k), H(k)* → *Y(k) = X(k).H(k)*)

Linear convolution using the DFT

- Procedure ($N = L + M - 1$)
  1. a. Pad $x(n)$ of length L samples with at least $M-1$ zeros
     b. Calculate $X(k)$ = DFT [$x(n)$] in N points, $k = 0, \ldots, N-1$
  2. a. Pad $h(n)$ of length M samples with at least $L-1$ zeros
     b. Calculate $H(k)$ = DFT [$h(n)$] in N points, $k = 0, \ldots, N-1$
  3. Multiply, $Y(k) = X(k) \cdot H(k)$ in N points, $k = 0, \ldots, N-1$
  4. Calculate $y(n)$ = IDFT [$Y(k)$] in N points, $n = 0, \ldots, N-1$

(If $N = L + M - 1 \rightarrow$ No aliasing)



(Additional materials)

- DTFT and MATLAB
- DTFT example
- DFT in matrix interpretation
- Bandwidth of frequency domain samples
- Inverse DFT (IDFT)
- IDFT in matrix interpretation
- Convolution
- Overlap-add convolution

DFT and MATLAB

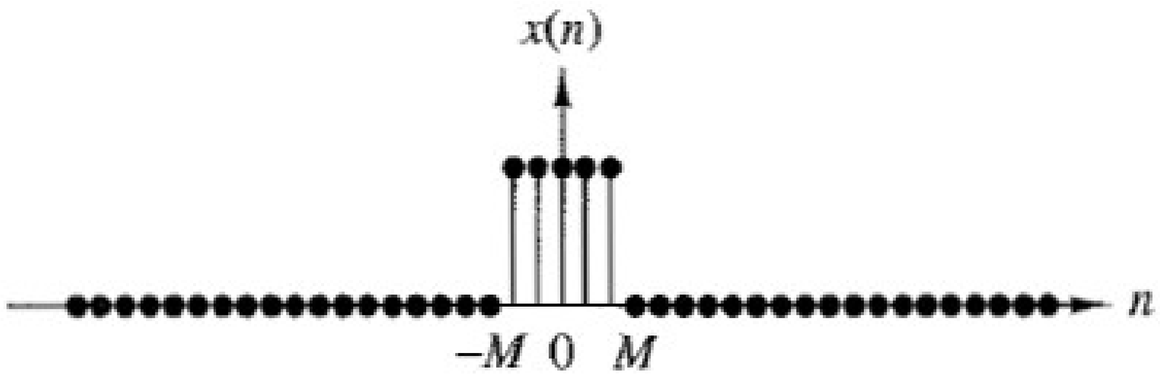
- DFT in MATLAB
  - The DFT is used to sample DTFT of signal $x(n)$ on an arbitrary grid
- X = freqz(x, 1, w);
  - Samples the DTFT of signal $x(n)$ at frequencies in $w(k)$
- X = fft(x);
  - Derives the DFT in N points of a signal of length N samples

DTFT example

- Derive DTFT of the following signal

$$x(n) = A.u(n+M) - A.u(n-M+1)$$

$$A = 1,\ M = 2,\ N = 2.M+1 = 5$$



DTFT example

- Derive DTFT of the following signal

$$x(n) = A.u(n+M) - A.u(n-M+1)$$

$$A = 1,\ M = 2,\ N = 2.M+1 = 5$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=-M}^{M} A.e^{-j\omega n}$$

$$X(e^{j\omega}) = A\frac{e^{j\omega M} - e^{-j\omega(M+1)}}{1-e^{-j\omega}} = A\frac{\sin\left(\left(M+\frac{1}{2}\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)},\ \omega\neq 0$$

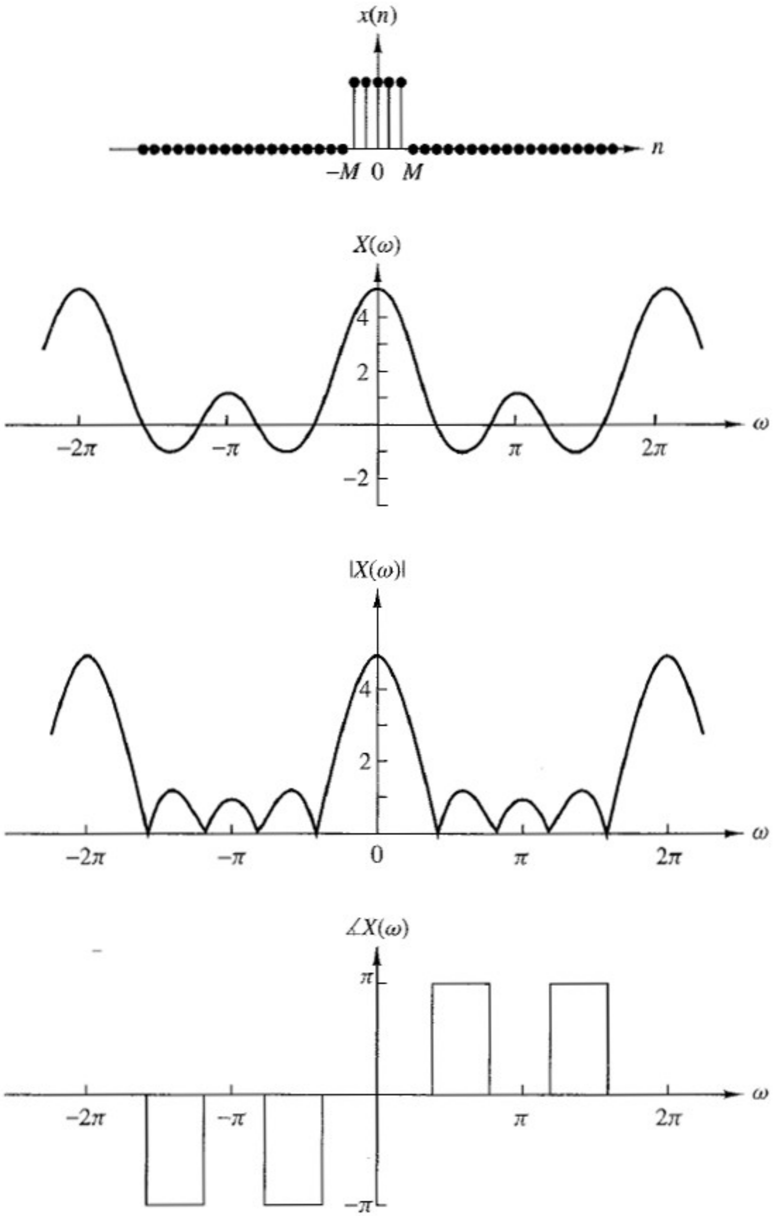
$$X(e^{j\omega}) = A.(2M+1),\quad \omega=0$$

$$\left|X(e^{j\omega})\right| = |A|\left|\frac{\sin\left(\left(M+\frac{1}{2}\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)}\right|,\ \omega\neq 0$$

$$\left|X(e^{j\omega})\right| = |A|(2M+1),\quad \omega=0$$

$$\theta(e^{j\omega}) = \measuredangle X(e^{j\omega}) = 0,\ \pi \qquad X(e^{j\omega}) > 0$$

$$\theta(e^{j\omega}) = \measuredangle X(e^{j\omega}) = 0,\ -\pi,\ \ X(e^{j\omega}) < 0$$



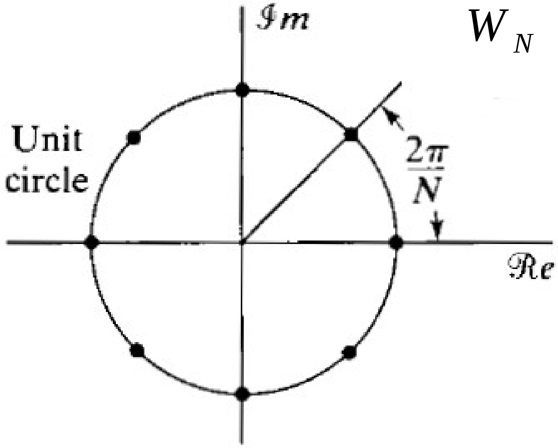
DFT in matrix interpretation

$$\boldsymbol{X}_N = \boldsymbol{W}_N \cdot \boldsymbol{x}_N$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \cdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-2)} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ W_N^0 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \cdots \\ x(N-1) \end{bmatrix}$$

Roots of the unit circle

$$W_N = e^{-j\frac{2\pi}{N}}$$



$$W_N^0 = 1, \quad W_N^N = 1$$

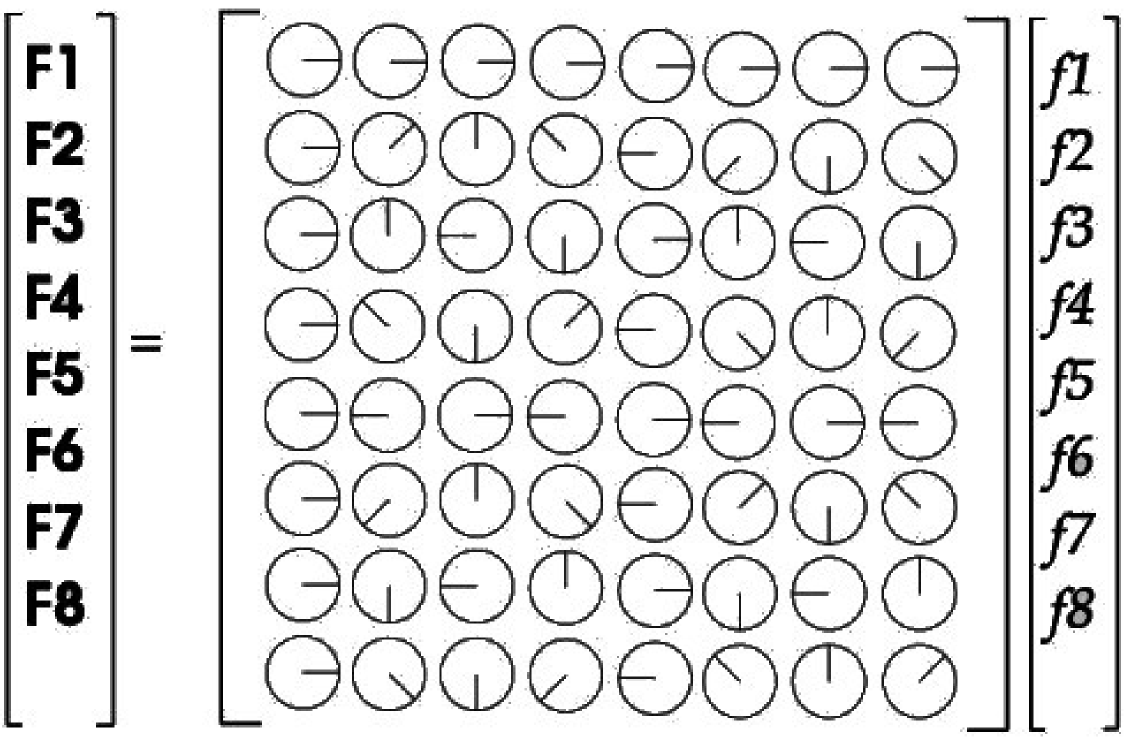
$$W_N^k \neq 1, \quad k = 1, 2, \ldots, N-1$$

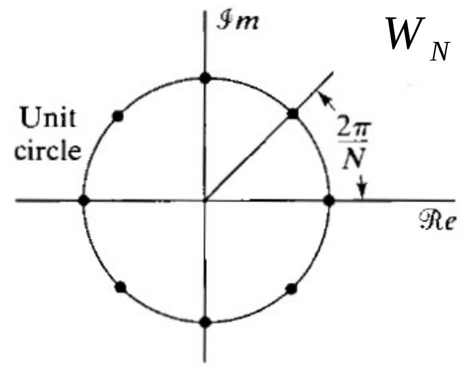
$$W_N^{k+N} = W_N^k \quad \text{Periodicity}$$

$$W_N^{k+N/2} = -W_N^k \quad \text{Symmetry}$$

DFT in matrix interpretation

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k}{N}.n}, \quad N=8$$





$$W_N = e^{-j\frac{2\pi}{N}}$$

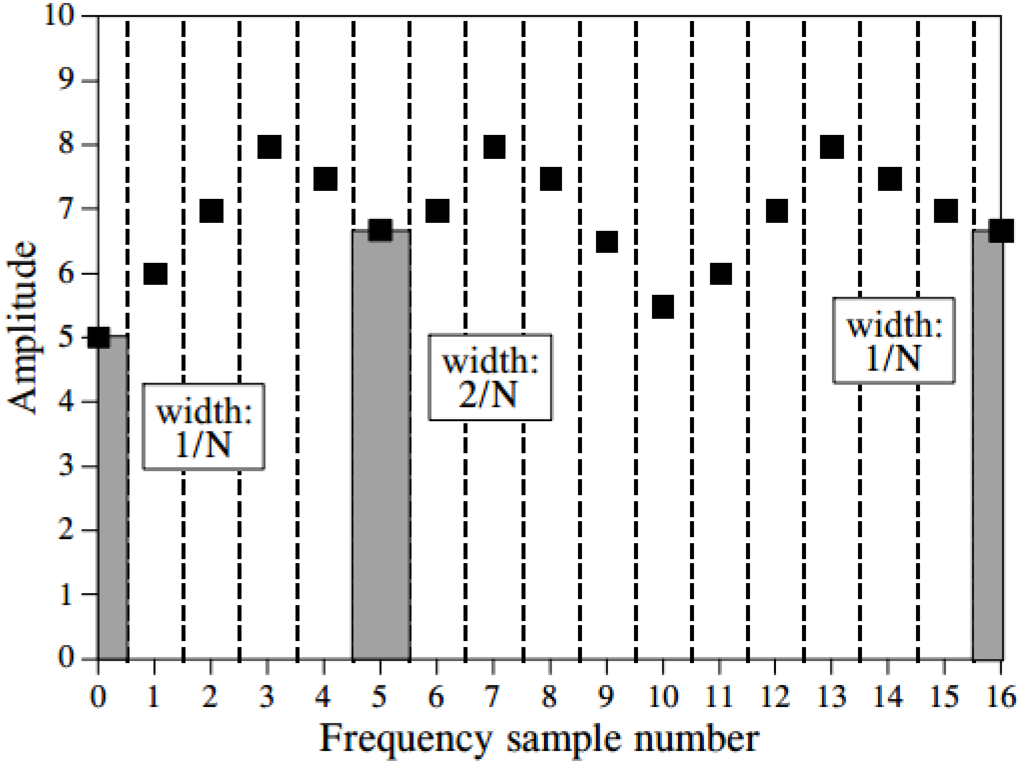
$$W_N^0 = 1, \quad W_N^N = 1$$

$$W_N^k \neq 1, \quad k=1,2,\dots,N-1$$

$$W_N^{k+N} = W_N^k \quad \text{Periodicity}$$

$$W_N^{k+N/2} = -W_N^k \quad \text{Symmetry}$$

Bandwidth of frequency domain samples



Inverse DFT (IDFT)

- Inverse DFT (IDFT)

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi n}{N}.k}$$

→ finite, discrete and periodic with a period of N

- If $W_N = e^{-j\frac{2\pi}{N}}$

$$x(n) = \sum_{k=0}^{N-1} X(k)\, W_N^{-nk}$$

- Verification

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{l=0}^{N-1} x(l)\, W_N^{kl}\right) W_N^{-nk}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} x(l) \sum_{k=0}^{N-1} W_N^{k(l-n)} = x(n)$$

IDFT in matrix interpretation

$$\boldsymbol{x}_N = \boldsymbol{W}_N^{-1} \cdot \boldsymbol{X}_N$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \cdots \\ x(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ W_N^0 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ W_N^0 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-2)} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ W_N^0 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \cdots \\ X(N-1) \end{bmatrix}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

Convolution

- Verification that convolution in time domain becomes multiplication in frequency domain

$$y(n) = x(n) * h(n)$$

$$\Rightarrow \quad Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (x(n)*h(n)) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \Big(\sum_{k=-\infty}^{\infty} x(k) h(n-k) \Big) e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} \Big(x(k) e^{-j\omega k} \sum_{n=-\infty}^{\infty} h(n-k) e^{-j\omega(n-k)} \Big)$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} \big(x(k) e^{-j\omega k} \big)}_{X(e^{j\omega n})} \underbrace{\sum_{n=-\infty}^{\infty} \big(h(n-k) e^{-j\omega n} \big)}_{(e^{-j\omega k}) H(e^{j\omega n})} (e^{j\omega k})$$

$$= X(e^{j\omega n}) \quad (e^{-j\omega k}) H(e^{j\omega n}) \quad (e^{j\omega k})$$

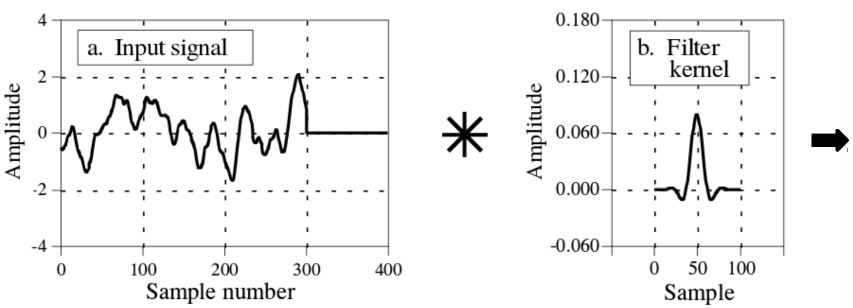
$$Y(e^{j\omega}) = X(e^{j\omega}) . H(e^{j\omega})$$

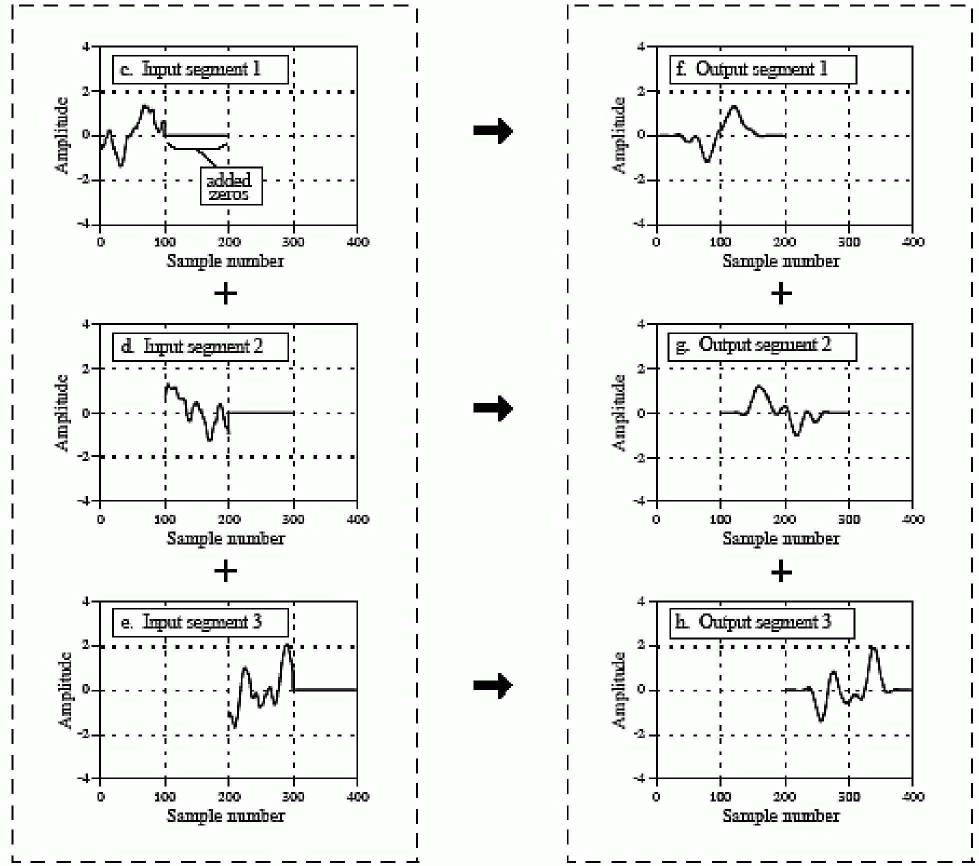
$$\text{DTFT}[\, x(n)*h(n) \,] = X(e^{j\omega}) \, . \, H(e^{j\omega})$$

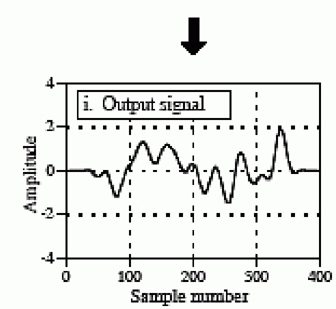
$$x(n)*h(n) \leftrightarrow X(e^{j\omega}) \, . \, H(e^{j\omega})$$

Overlap-add convolution

- If $x(n)$ is very long (real-time case), and $h(n)$ is of finite duration M







1. Decompose input signal, $x(n)$, into segments of length L
2. Pad each segment with at least M zeros to allow the expansion during the convolution
3. Convolve each of the input segments with $h(h)$ of using DFT to produce the output segments
4. The output signal, $y(n)$, is found by adding the overlapping output segments
   → *The result is linear convolution*

[GDSP]