Discrete-time signals and systems, III

- Discrete-time systems described by Linear Constant-Coefficient Difference Equations (LCCDE)
- Finite Impulse Response (FIR) systems
- Infinite Impulse Response (IIR) systems
- Correlation
- Autocorrelation
- (Additional materials)

Discrete-time systems described by Linear Constant-Coefficient Difference Equations (LCCDE)

- An important class of discrete-time LTI systems expressed by Linear Constant-Coefficient Difference Equation (LCCDE)
- General specification of discrete-time LTI systems

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)\,, \qquad a_0 = 1$$

- Rearrange for $y(n)$ in causal form

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

- The output $y(n)$ is found by using M input data observed up to now and N output data already found before (*order* = max (N, M))
- The initial conditions are required (if they are assumed to be zero, system is linear, time-invariant and causal)

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Discrete-time systems described by Linear Constant-Coefficient Difference Equations (LCCDE)

$$y(n) = -\sum_{k=1}^{N} a_k \, y(n-k) + \sum_{k=0}^{M} b_k \, x(n-k)$$

- Example
  - Moving average
  - Difference equation representation

$$y(n) = y(n-1) + \frac{1}{3} x(n) - \frac{1}{3} x(n-3)$$

$$N = 1, \ a_1 = -1, \quad M = 3, \ b_0 = \frac{1}{3}, \ b_1 = 0, \ b_2 = 0, \ b_3 = -\frac{1}{3}$$

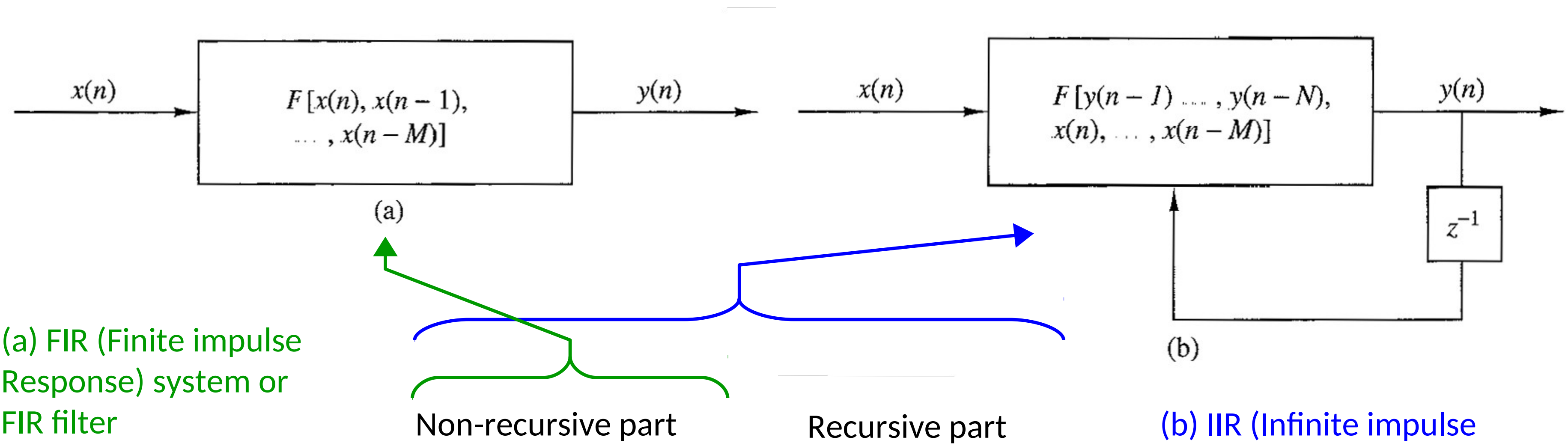
Discrete-time systems described by Linear Constant-Coefficient Difference Equations (LCCDE)

- Why Linear Constant-Coefficient Difference Equations (LCCDE) ?
  - Convolution can be used to realize only those LTI systems for which the impulse response *h(n)* is of finite duration
  - Representing an LTI system in terms of impulse response is not easy if the length of *h(n)* is infinite
  - Realization of the system is easily observed from the LCCDE equation
  - The output can be found using finite number of computations
  - Different kinds of digital signal processors (filters) can be designed by selecting properly the coefficients

$$y(n) = -\sum_{k=1}^{N} a_k \, y(n-k) + \sum_{k=0}^{M} b_k \, x(n-k)$$

Discrete-time systems described by Linear Constant-Coefficient Difference Equations (LCCDE)

- (a) non-recursive and (b) recursive discrete-time LTI systems



(a) FIR (Finite impulse Response) system or FIR filter

(b) IIR (Infinite impulse Response) system or IIR filter

$$y(n) = \sum_{k=0}^{M} b_k\, x(n-k) - \sum_{k=1}^{N} a_k\, y(n-k)$$

[Proakis, Manolakis]

Finite Impulse Response (FIR) systems

- FIR (non-recursive) system
- The LCCDE equation contains non-recursive term only, $a_k = 0$ (k=1,2,3,...)

$$y(n) = \sum_{k=0}^{M} b_k\, x(n-k) - \cancel{\sum_{k=1}^{N} a_k\, y(n-k)}$$

- Properties
  - Impulse response, $h(n)$, of finite duration, defined simply by b_k coefficients
  - Always stable (no feedback connection)
  - Larger number of coefficients needed
  - Linear phase (all frequencies are delayed for the same amount)
  - Implementation using convolution sum possible, $b_k = h(k)$, k=0,1,2,...,N
- Examples, simple lowpass filter, $y(n) = 1/2\, x(n) + 1/2\, x(n-1)$

  simple highpass filter, $y(n) = 1/2\, x(n) - 1/2\, x(n-1)$

  weighted average, $y(n) = 0.7\, x(n) + 0.9\, x(n-1) + 0.3\, x(n-2)$

Infinite Impulse Response (IIR) systems

- IIR (recursive) system
- The LCCDE equation contains recursive term, at least one $a_k \neq 0$

$$y(n) = \sum_{k=0}^{M} b_k \, x(n-k) - \sum_{k=1}^{N} a_k \, y(n-k)$$

- Properties

  - Impulse response, *h(n)*, of infinite duration
  - Feedback connection
  - Can be unstable, e.g., $y(n) = 1.5\, y(n-1)$
  - Non-linear phase (all frequencies are delayed and shifted among each other)
  - Lower number of coefficients needed
- Example

$$y(n) = x(n) - 0.7\, x(n-1) + 0.4\, y(n-1)$$

Infinite Impulse Response (IIR) systems

- Example of IIR system (Also an example of exam task)

$$y(n) = a.y(n-1) + x(n)$$

1) Sketch a block diagram for the following difference equation.
2) Derive the first five values of the impulse response and of the unit step response of the filter.
3) Is this filter with finite of infinite impulse response?
4) Sketch the impulse response and unit step response for $a = 0.5$.
5) Check stability of the filter. Is this filter stable or unstable?
6) For what values of a the filter is unstable?

Correlation

- Correlation identifies similarity between two signals
- Cross-correlation of *x(n)* and *y(n)*

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-l) = \sum_{n=-\infty}^{\infty} x(n+l)\, y(n) \qquad l=0,\pm1,\pm2,\ldots$$

- Note

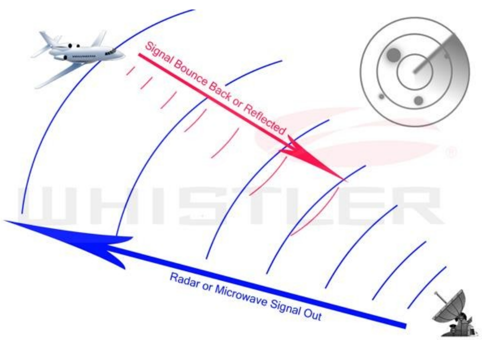
$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\, x(n-l) = \sum_{n=-\infty}^{\infty} y(n+l)\, x(n) = r_{xy}(-l)$$

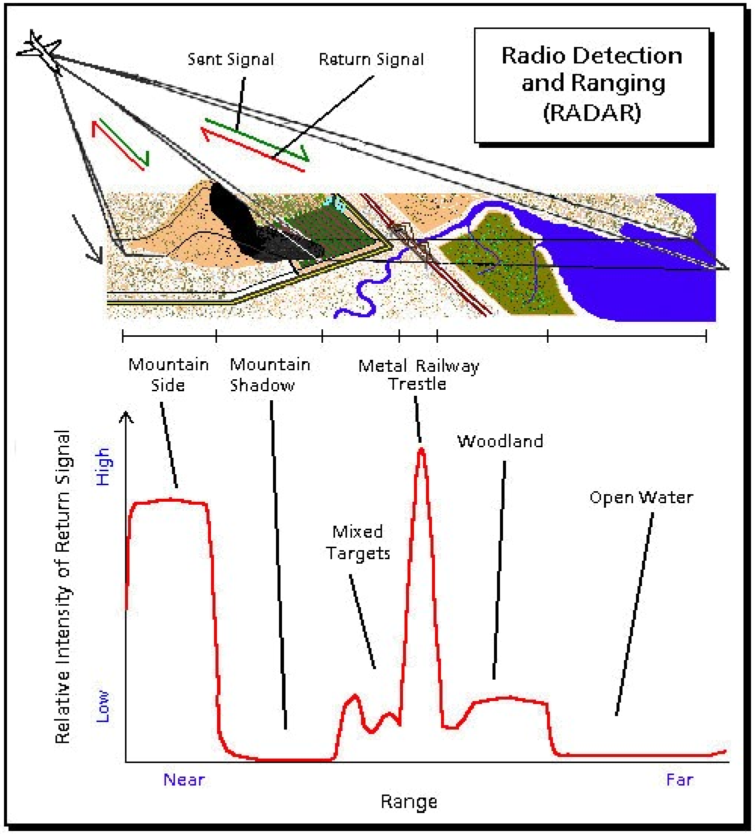
- The index *l* is the time shift or *lag*
- Cross-correlation is a measure of similarity of two signals *x(n)* and *y(n)* for different lags

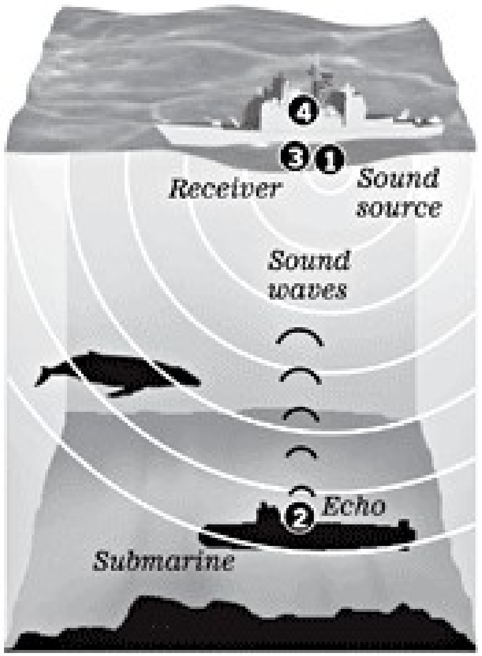
Convolution: $\sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

Correlation

- Many applications when the task is to detect / locate / measure a reference signal in some other signal
- E.g., Radar, sonar application

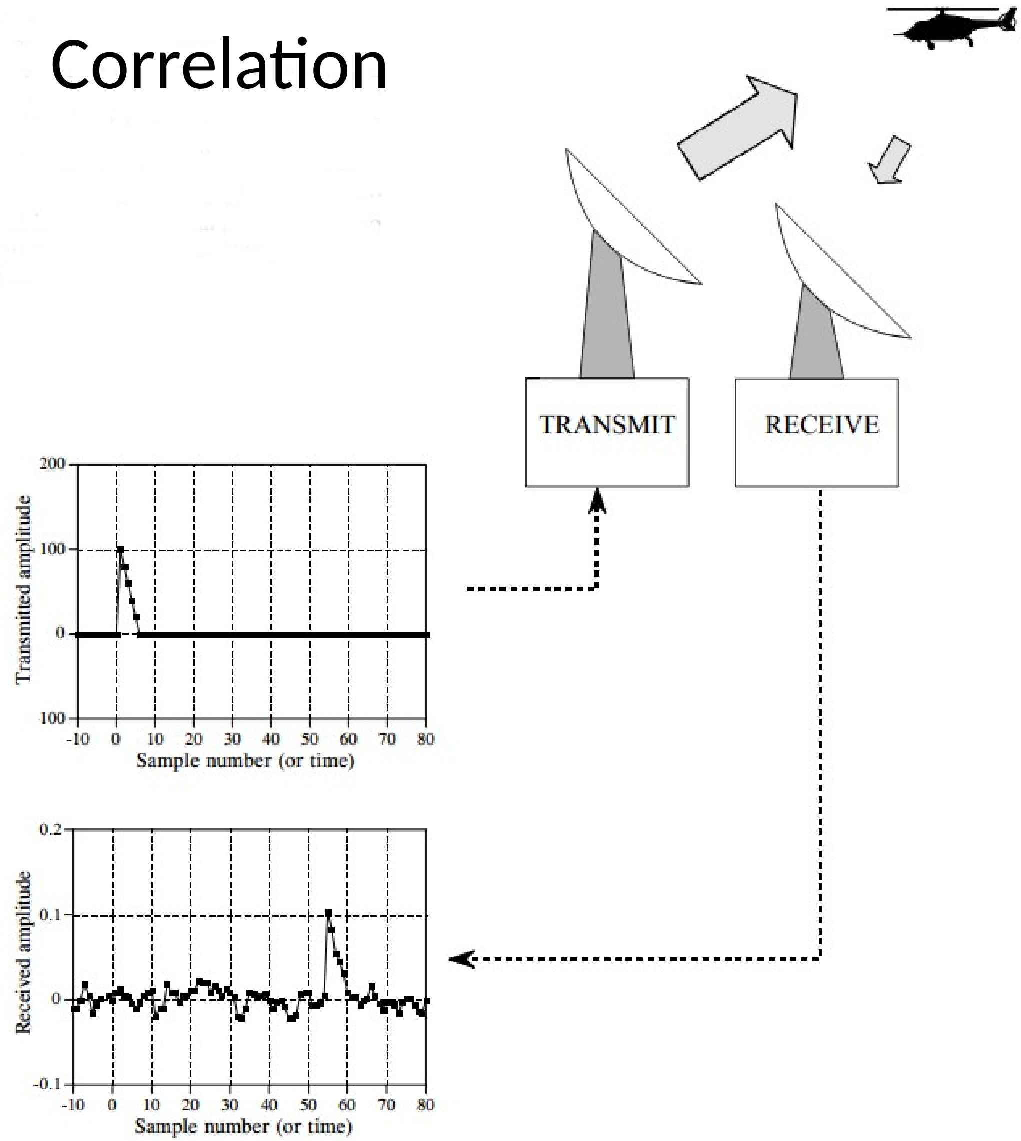






Correlation

- Radar, (sonar)



[GDSP]

Correlation

- Applications when the task is to detect / locate / measure a reference signal *t(n)* in some other signal *x(n)*
- Radar, sonar application
  - Let *t(n)* represents the transmitted (reference) signal and *x(n)* the received signal
  - If a target is present in the space, the received signal *x(n)* consists of a delayed version of the transmitted signal, reflected form the target, and corrupted by additive noise

$$x(n) = \alpha\, t(n-D) + w(n)$$

  - Correlation of *x(n)* and *t(n)*, $y_{xt}(l)$, provides us with information about the time delay *D* which defines the distance to the target

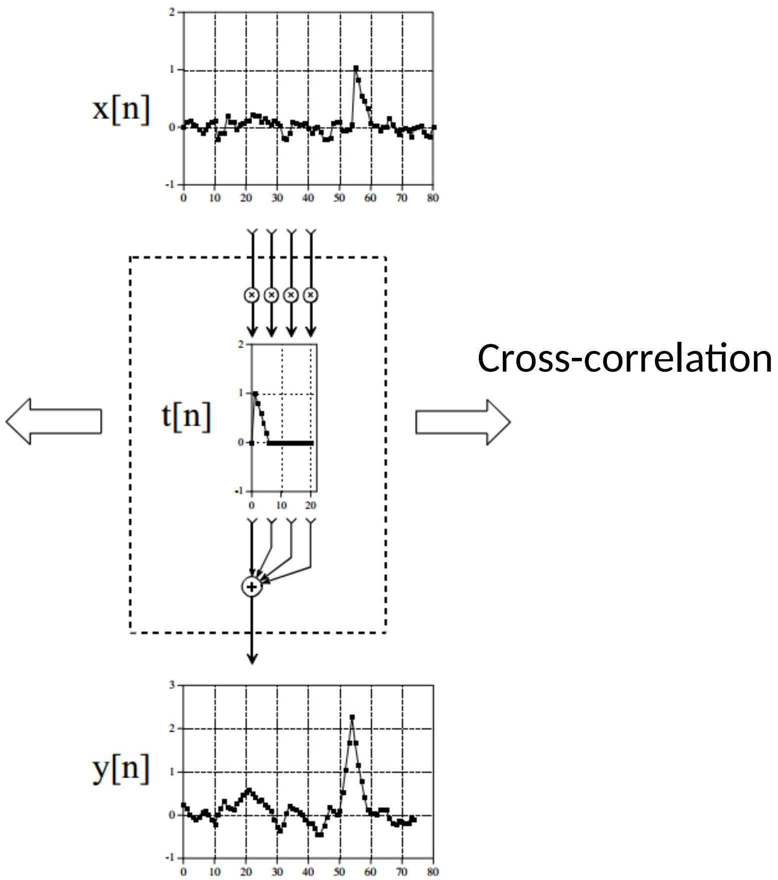
$$y_{xt}(l) = \sum_{n=-\infty}^{\infty} x(n)\, t(n-l)$$

Correlation

- Radar, (sonar)

Measuring similarity for different lags between *t(n)* – transmitted signal and *x(n)* – received signal

Position of local maximum defines the time delay *D*



[GDSP]

Correlation

- Many applications when the task is to detect / locate /measure a reference signal in some other signal
  - Detection of amplitude and estimating the delay of reference signal (echo detection, echo detection and cancellation)
  - Sound detection (detecting and locating sounds)
  - Biological signal processing (detecting events, heartbeat detection)
  - Image processing (character recognition)
  - ...

Autocorrelation

- Autocorrelation is correlation of signal with itself

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\, x(n-l), \qquad l = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$r_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2(n) \rightarrow \text{energy of signal } x(n)$$

- The index l is the time shift or *lag*
- Autocorrelation is a measure of similarity of signal $x(n)$ with itself for different lags

Autocorrelation

- Many applications when the task is to estimate periodicity (period) of a signal, or, to detect the presence of a periodic signal in noise
- Simple example

$x(n) = \{ 1, 0, -1, 0, 1, 0, -1, 0 \}$ (period of four samples)

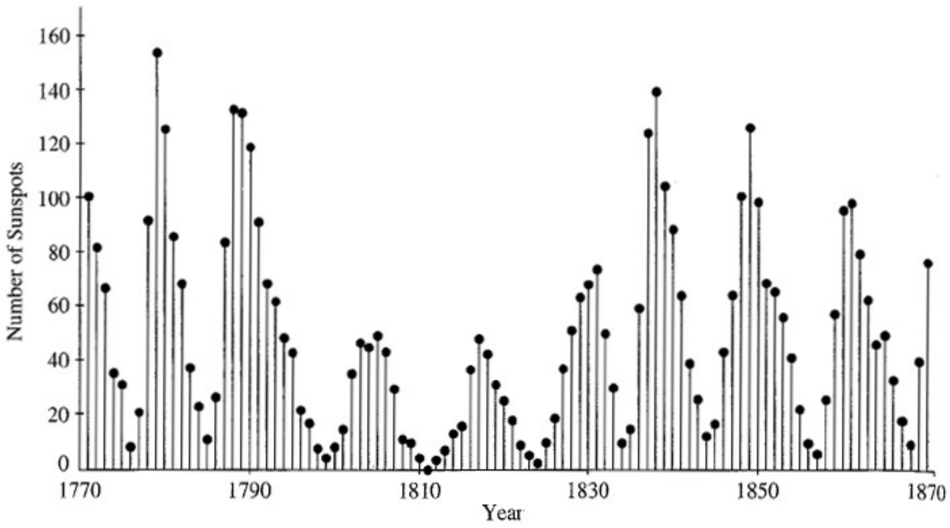
↑ $n = 0$ ↑ $n = 4$

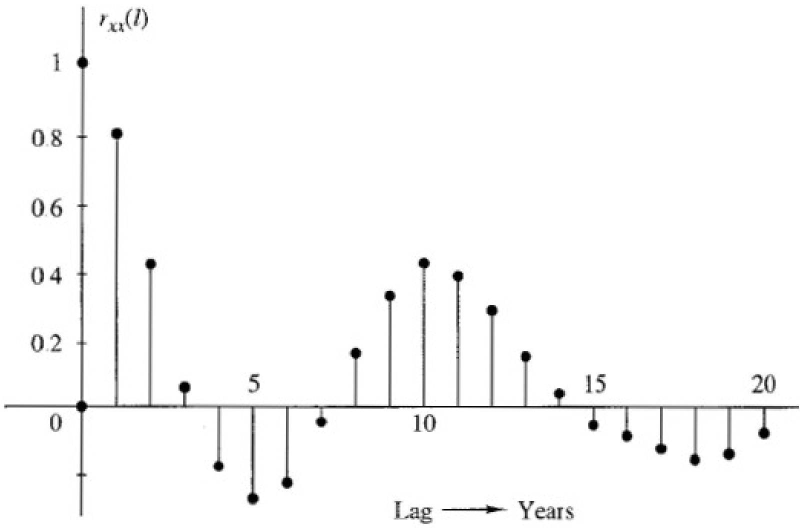
$r_{xx}(l) = \{ 0, -1, 0, 2, 0, -3, 0, 4, 0, -3, 0, 2, 0, -1, 0 \}$ (extrema are four samples apart)

↑ $l = -4$ ↑ $l = 0$ ↑ $l = 4$

Autocorrelation

- Identification of periodicity in sunspot numbers





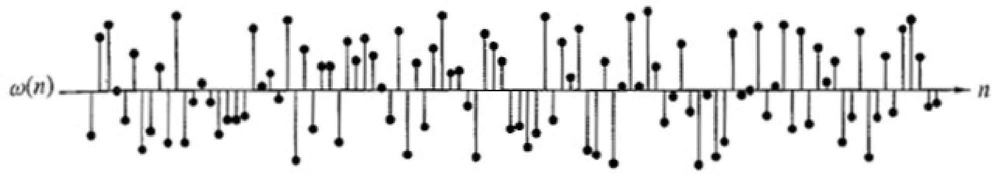
Periodicity with a period of 10 to 11 years.

[Proakis, Manolakis]

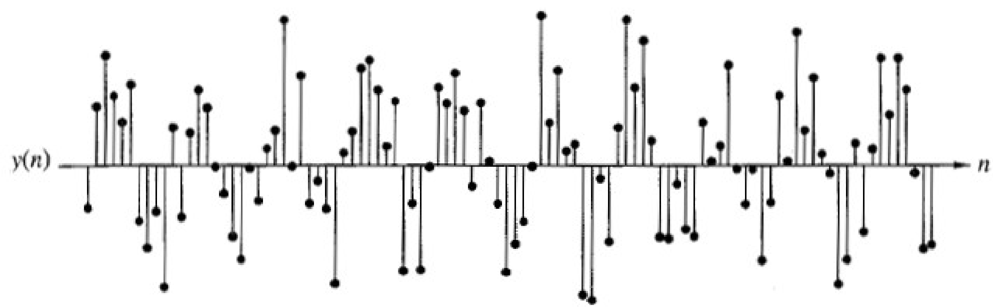
Autocorrelation

- Detecting the presence of a periodic signal, *x(n)*, corrupted by noise

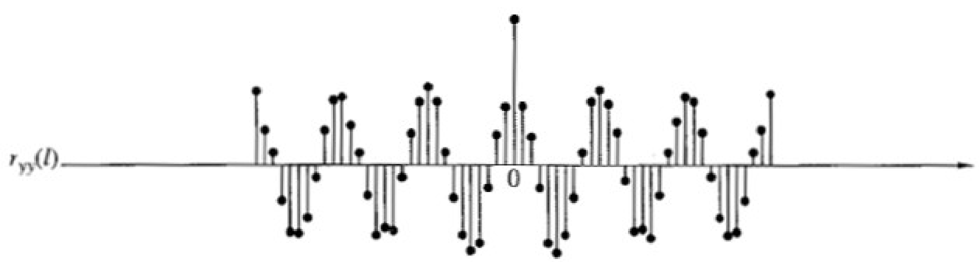
Noise, *w(n)*



Signal and noise,
y(n) = x(n) + w(n)



Autocorrelation,
$r_{yy}(l)$



[Proakis, Manolakis]

Autocorrelation

- Many applications when the task is to estimate periodicity (period) of a signal, or, to detect the presence of a periodic signal in noise
  - Sound recognition
  - Voice recognition
  - Melody recognition
  - Noise rejection (rising of Signal to Noise Ratio - SNR)

(Additional materials)

- Finite Impulse Response (FIR) systems
- Infinite Impulse Response (IIR) systems
- Correlation

Finite Impulse Response (FIR) systems

- More examples of FIR systems

1. Simple gain, or amplifier:

$$y[n] = Gx[n]$$

2. Delay of n_0 samples:

$$y[n] = x[n - n_0]$$

3. Two-point moving average:

$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$

4. Euler's formula for approximating the derivative of a continuous-time function:

$$y[n] = \frac{x[n] - x[n-1]}{T_s}$$

   where T_s is the sampling interval.

5. Averaging over N consecutive epochs of duration L:

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n - kL]$$

Finite Impulse Response (FIR) systems

- More examples of FIR systems (impulse responses)

1. Gain:

$$h[n] = G\delta[n]$$

2. Delay:

$$h[n] = \delta[n - n_0]$$

3. Two-point moving average:

$$h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$$

4. Euler's approximation to the derivative:

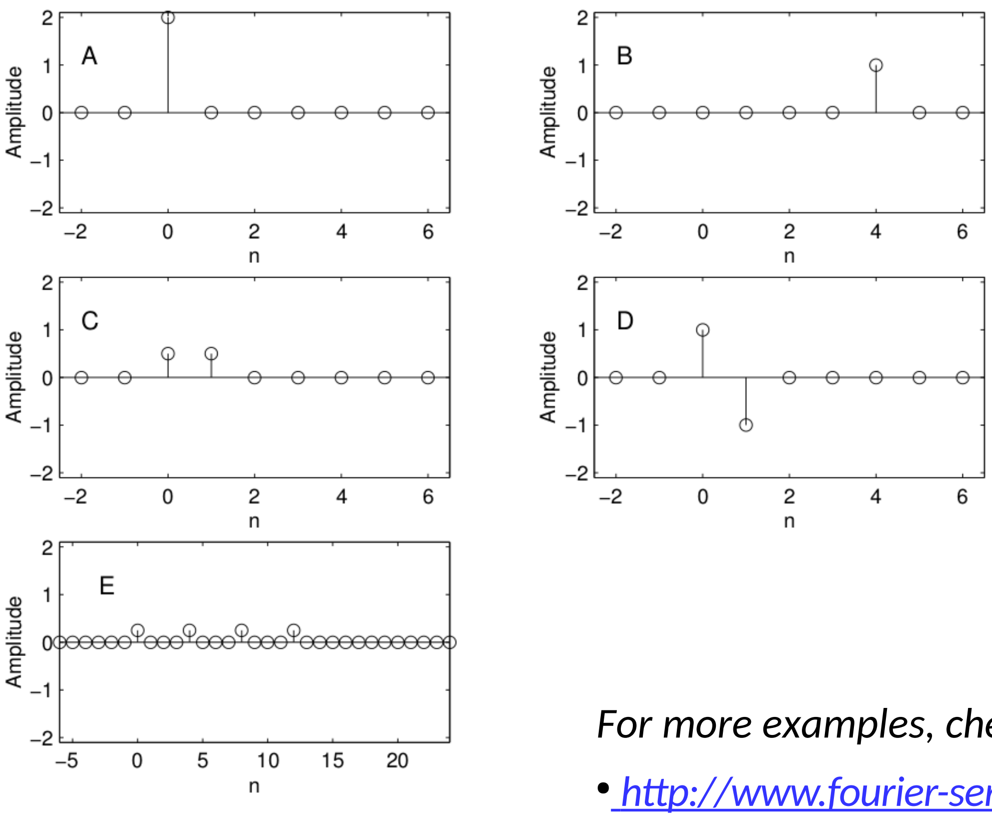
$$h[n] = (\delta[n] - \delta[n-1])/T_s$$

5. Averager:

$$h[n] = \frac{1}{N}\sum_{k=0}^{N-1}\delta[n - kL]$$

Finite Impulse Response (FIR) systems

- More examples of FIR systems (impulse responses)



For more examples, check also

- http://www.fourier-series.com/

Infinite Impulse Response (IIR) systems

- Examples of IIR systems

6. Trapezoidal integration formula:

$$y(n) = a\,y(n-1) + \frac{(x(n) + x(n-1))T_S}{2}$$

7. Digital “leaky integrator”, of first-order lowpass filter:

$$y(n) = a\,y(n-1) + x(n) \qquad 0 < a < 1$$

Correlation

- Cross-correlation may be calculated as convolution with time-reversed signal

Correlation

$$r_{xy}(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(k-n)$$

Convolution

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)$$

$\longrightarrow$

$$r_{xy}(n) = x(n) * y(-n)$$