Discrete-time signals and systems, I

- Continuous-time sinusoidal signals
- Signals
- Conversion to discrete signal
- Conversion to digital signal
- Types of signals
- Sampling a sinusoid
- The Nyquist sampling theorem
- Representation of discrete signals
- Basic signals
- Properties of discrete signals
- Operations on discrete signals
- Combination of basic operations
- (Additional materials)

Continuous-time sinusoidal signals

- Cosine signal

$$x_a(t) = A\cos(\Omega t + \theta), \qquad -\infty < t < \infty$$

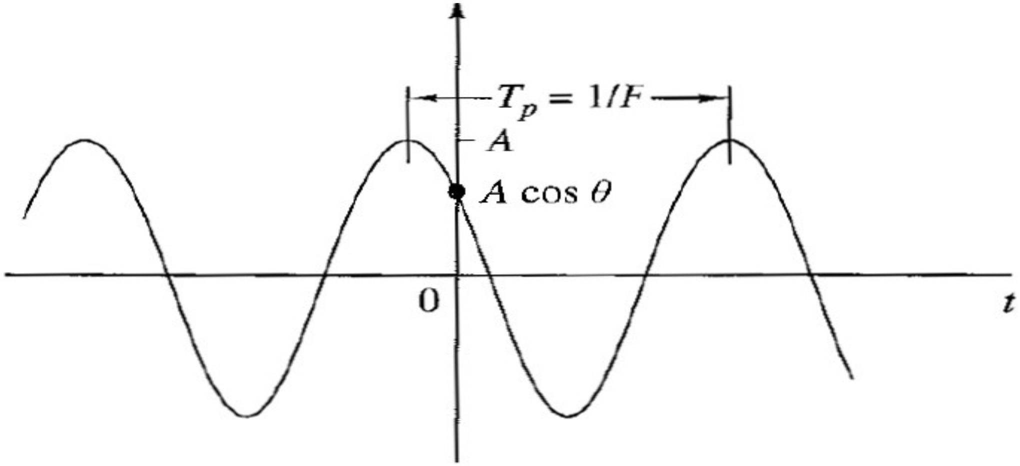
A is the amplitude

Ω is the frequency in radians per second [*rad/s*], $\Omega = 2\pi F$

Θ is the phase in radians [*rad*]

Tp is the duration of one cycle in seconds [*s*]

$F = 1 / Tp$ is the frequency in cycles per second or Hertz [*Hz*], $Hz = 1/s$

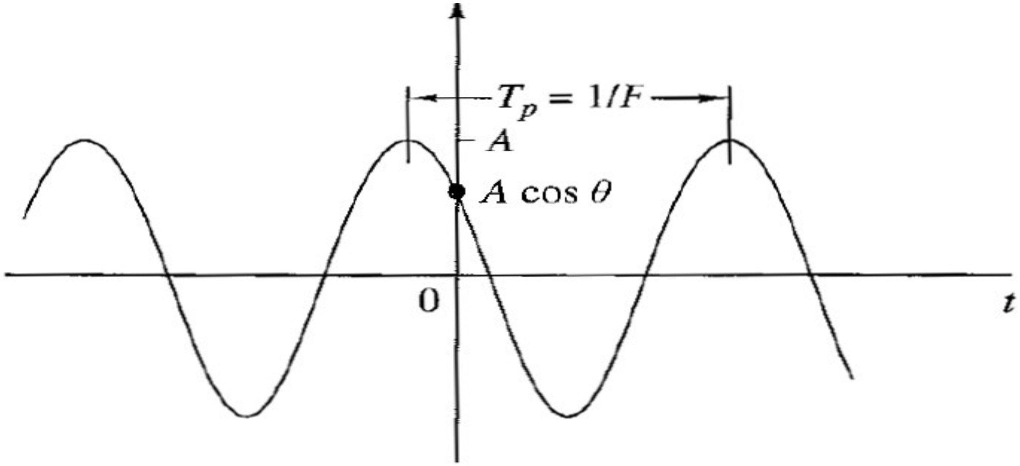


Continuous-time sinusoidal signals

- Cosine signal

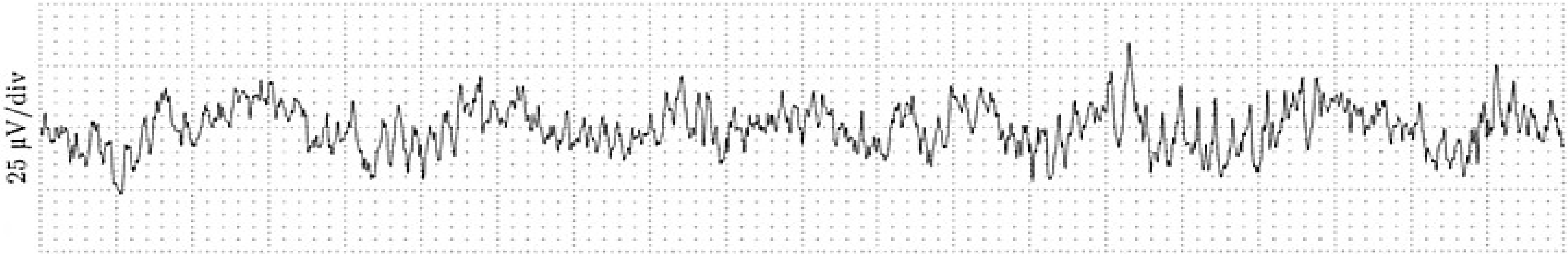
$$x_a(t) = A\cos(\Omega t + \theta), \qquad -\infty < t < \infty$$

- sin (30^0) = sin ($\pi / 6$) = 1 / 2 = cos (60^0) = cos ($\pi / 3$)
- sin (45^0) = sin ($\pi / 4$) = sqrt(2) / 2 = cos (45^0) = cos ($\pi / 4$)
- sin (60^0) = sin ($\pi / 3$) = sqrt(3) / 2 = cos (30^0) = cos ($\pi / 6$)
- cos (ϕ) = sin ($\phi + 90^0$) , sin (ϕ) = cos ($\phi - 90^0$)



Signals

- An electroencephalogram (1-Dimension)



- A segment of signal may be represented as a sum of several sinusoids of different amplitudes and frequencies:

$$\sum_{i=1}^{N} A_i(t)\sin(2\pi F_i(t)t + \theta_i(t))$$

where {*A*i*(t)*}, {*F*i*(t)*}, and {*Θ*i*(t)*} are the sets of amplitudes, frequencies and phases

- What is spectrum?

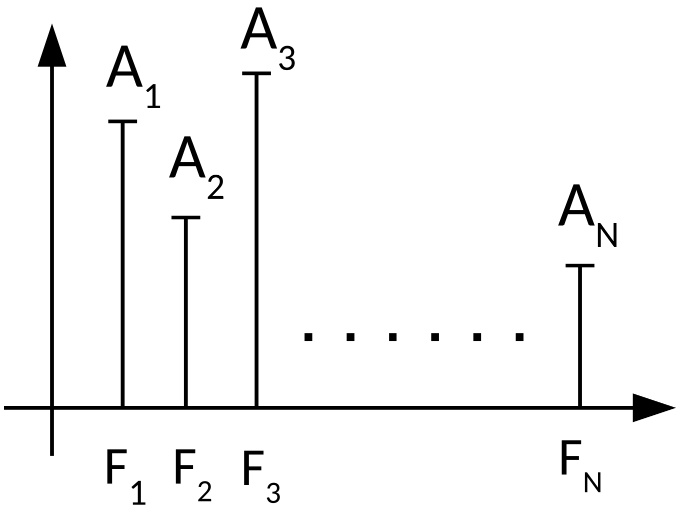
Signals

- A segment of signal may be represented as a sum of several sinusoids of different amplitudes and frequencies:

$$\sum_{i=1}^{N} A_i \sin(2\pi F_i t + \theta_i)$$

- where {*A*i}, {*F*i}, and {Θi} are the sets of amplitudes, frequencies and phases
- What is (frequency) spectrum?

Amplitude spectrum

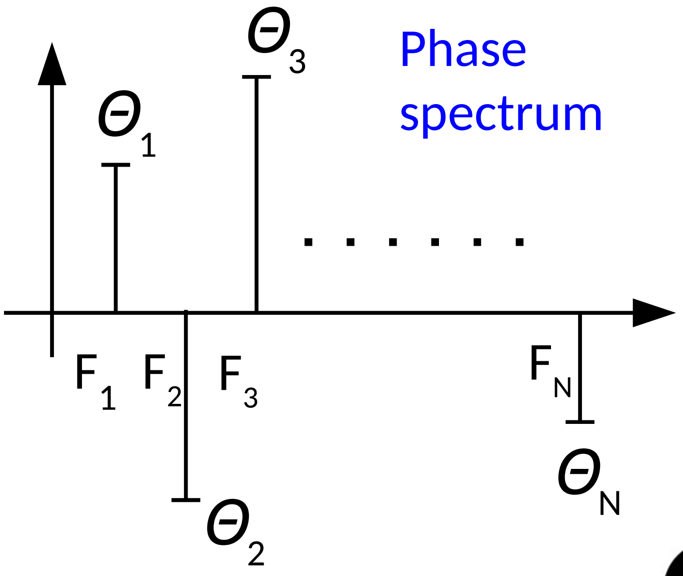


$F_2 = 2.F_1$

$F_3 = 3.F_1$

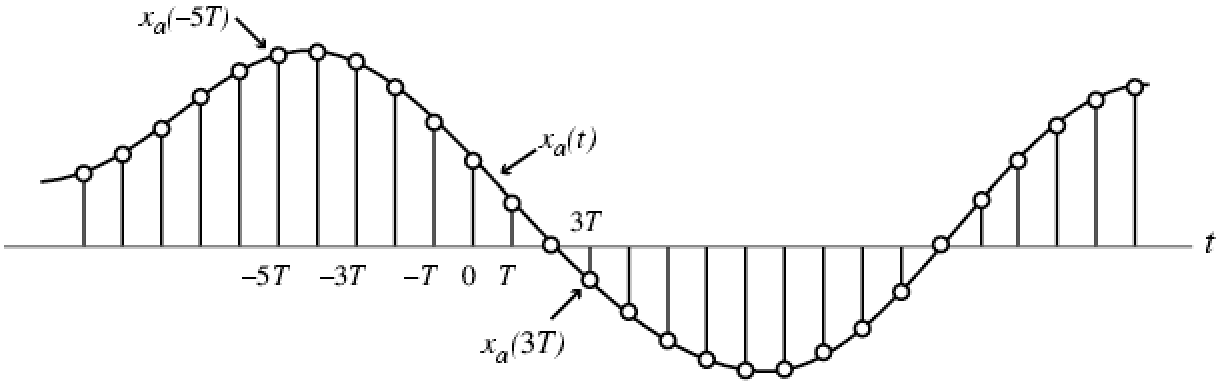
.
.
.

$F_N = N.F_1$


Phase spectrum

Conversion to discrete signal

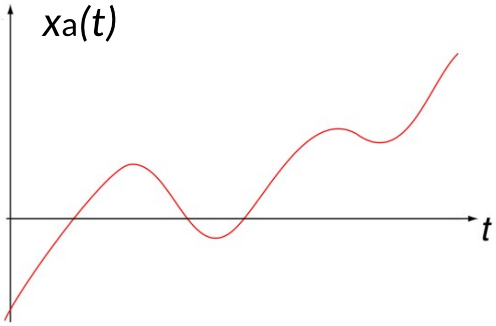
- Sampling
- Discrete signal (discrete-time signal), $x(n)$, is obtained by sampling a continuous-time signal, $x_a(t) = x(t)$, $x_a(t) \rightarrow x_a(nT) \rightarrow x(n)$



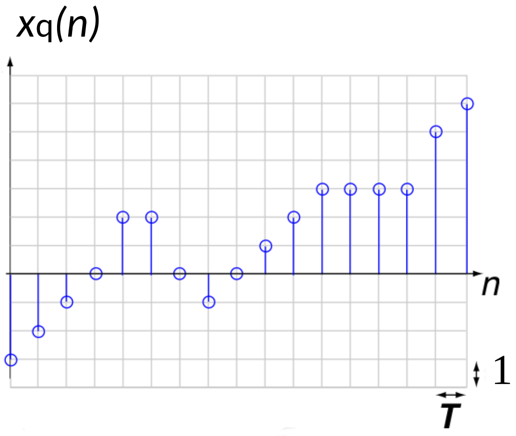
- Sequence of discrete signal samples, $\{x(n)\} = x_a(nT)$, $n = \ldots, -1, 0, 1, 2, \ldots$
- T (T_s) is sampling period or sampling interval in [s], [sec]
- $F_s = 1/T$ is sampling frequency or sampling rate in [smp/s], [smp/sec], [Hz]

Conversion to digital signal

- Sampling and quantization
- Digital signal, $x_q(n)$, is obtained by quantization of discrete signal samples, $x(n)$



$x_a(t) \rightarrow x(n) \rightarrow x_q(n)$



- The $x_q(n)$ is equal the closest integer
- Range of the values of quantized samples, $x_q(n)$, is following

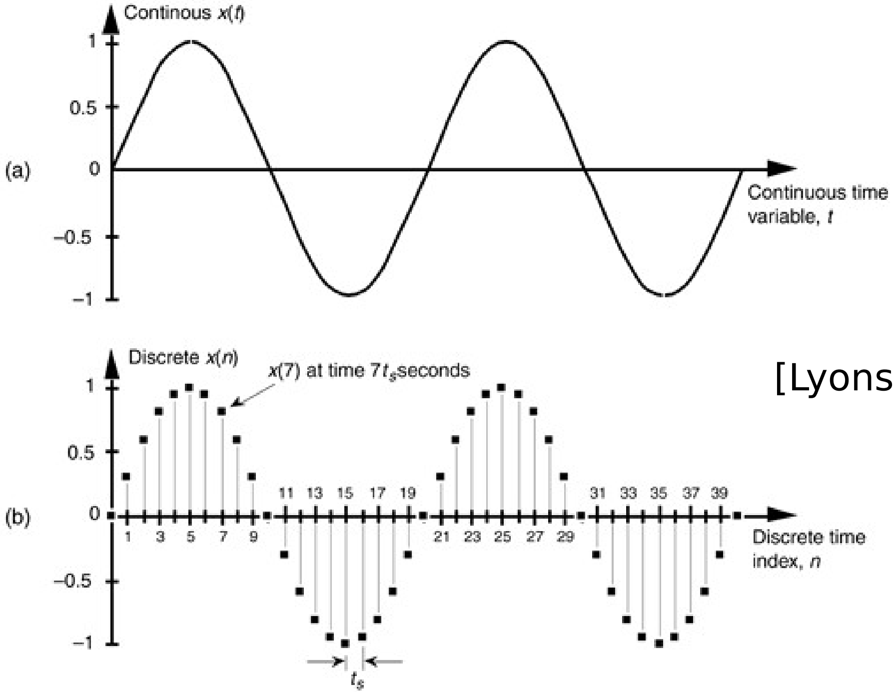
$$-x_{max} \le x_q(n) < x_{max}, \qquad x_{max} = 2^{M-1}$$

where M is the number of bits of quantizer

- Digital signal, $xq(n) \rightarrow x(n)$

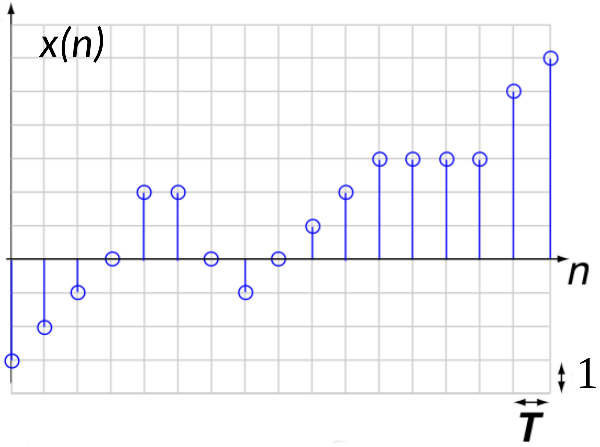
Types of signals

- Continuous signal, $x_a(t) = x(t)$ (a)
- Discrete signal, $x_a(nT) \rightarrow x(n)$ (b)
- Digital signal, $x_q(n) \rightarrow x(n)$



[Lyons]

Type of signal	Independent variable (time)	Range value
Continuos	Continuous	Continuous
Discrete	Discrete	Continuous
Digital	Discrete	Discrete



Sampling a sinusoid

- Discrete sinus signal, $x(n)$

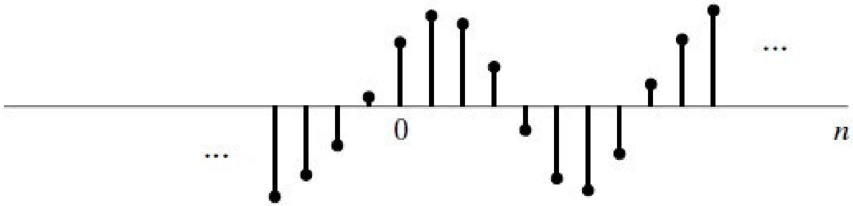
$$x_a(t) = A\sin(2\pi\, F\, t + \theta) \xrightarrow{\text{Sampling}} x_a(nT_s) = A\sin(2\pi\, F/F_s\, n + \theta) = x(n)$$

$$= A\sin(\Omega\, t + \theta) \qquad = A\sin(\omega\, n + \theta) = x(n)$$

$$\Omega = 2\pi\, F \longrightarrow \omega = 2\pi\, F/F_s \rightarrow \omega = 2\pi\, f$$

- F_s is sampling frequency in [*smp/s*] or in [*Hz*], $Hz = 1/s$

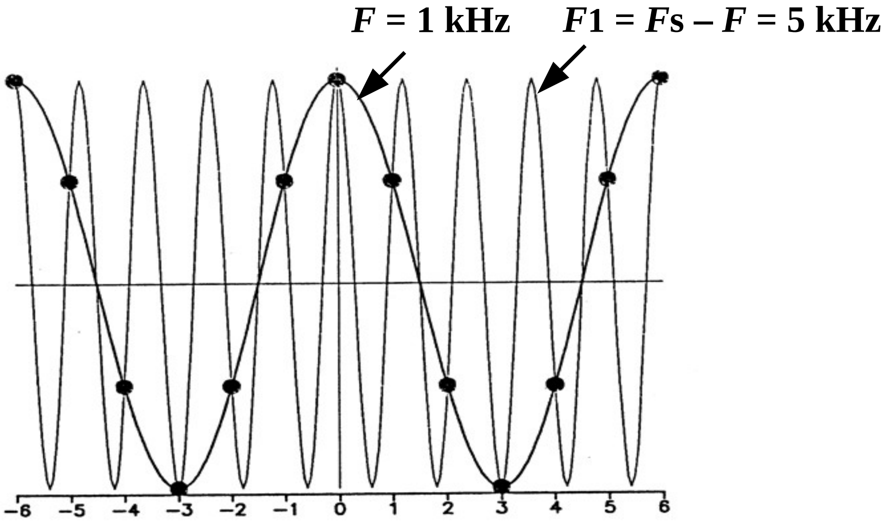
$$f = \frac{F}{F_s}$$



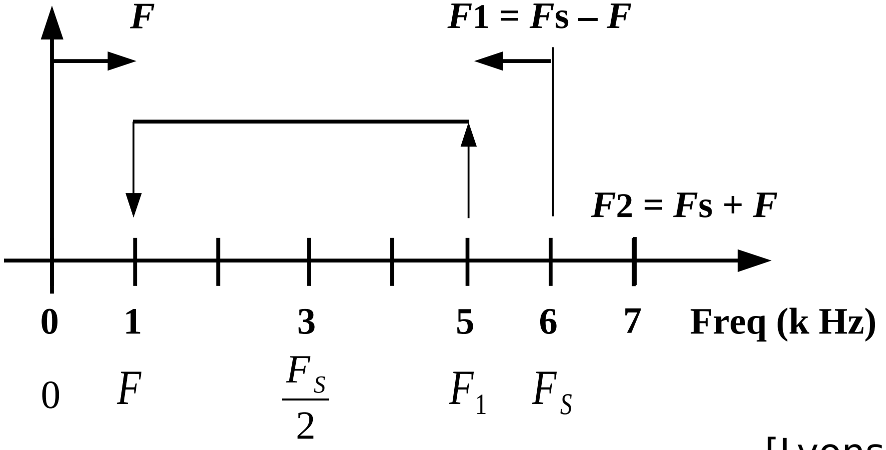
- *F* – *continuous-time frequency* in cycles per second [*Hz*]
- *f* - *discrete-time frequency* in cycles per sample [*cycles/sample*], [*cyc/smp*]

Sampling a sinusoid

- Aliasing
  - We say that a larger frequency appears aliased to a lower frequency
- *Fs* = 6 kHz



- It is not possible to know if the frequency of the original continuous-time signal *x(t)* was *F*, or, *F + Fs*, or, *F + 2Fs*, etc; or, *Fs – F*, or, *2Fs – F*, etc



  - How to avoid aliasing?
    - Regarding this example, what was the number of samples per sinusoid, *N*, that still approximated a sinusoid?

      N ≥ ?

[Lyons]

Sampling a sinusoid

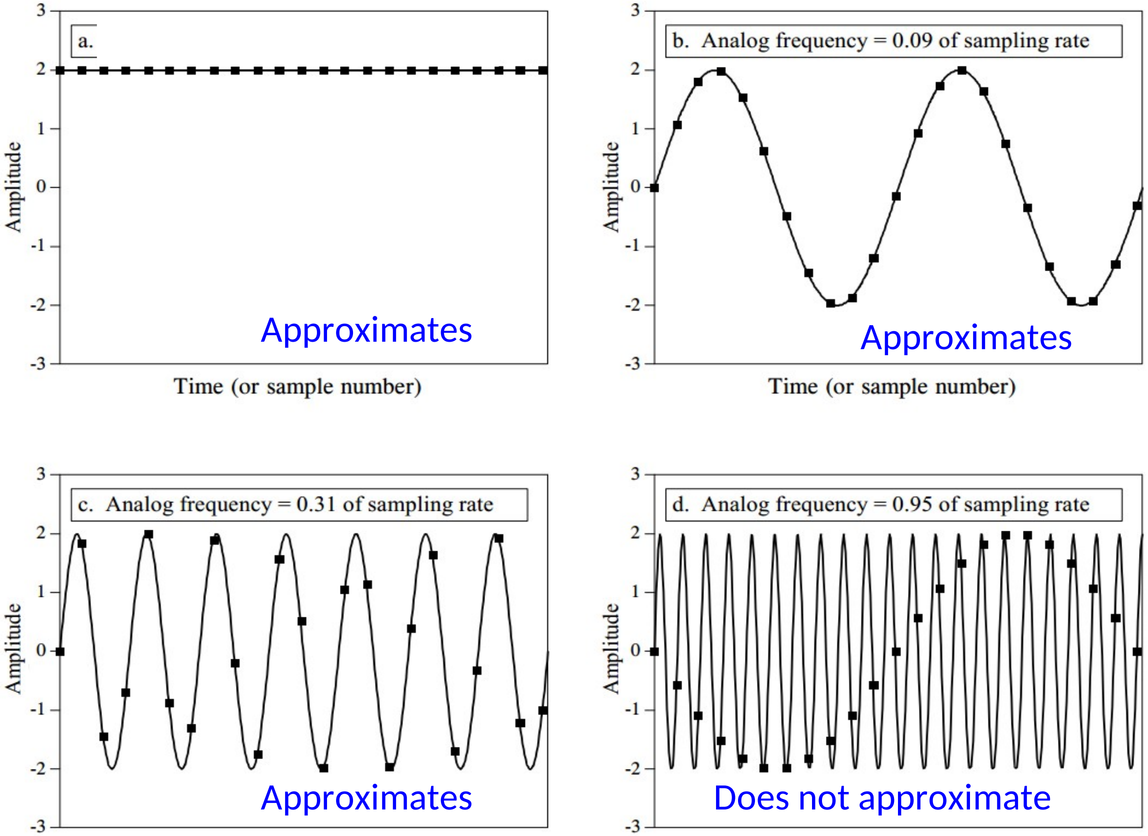
$$f = \frac{F}{F_S}$$

$$F = f\,F_S$$

- Aliasing

$x(n) = A \sin(2\pi\, F/Fs\; n), \quad A = 2$

$x(n) = 2$



Approximates

Approximates

Approximates

Does not approximate

$F = 0.09\ Fs$

$f = 0.09$

$N = 11.11$

N = number of samples per sinusoid

N = Fs / F = 1 / f

$F = 0.31\ Fs$

$f = 0.31$

$N = 3.23$

$F = 0.95\ Fs$

$f = 0.95$

$N = 1.05$

What N (or f) still approximates a sinusoid ?

[GDSP]

The Nyquist sampling theorem

- How to avoid aliasing?
- Regarding the previous example, what is the highest frequency F_{max} (expressed with F_s) of a sinusoid that would still be approximated, if using the sampling frequency F_s?

Since: $N \geq 2$ and $F_{max} / F_s \leq 0.5$, follows: $F_{max} \leq F_s / 2$

Answer: The highest frequency present in the input analog signal should be be less than or equal to $F_s / 2$, $F_{max} \leq F_s / 2$

If $F_s = 2.F_{max}$, the F_s is said to be Nyquist frequency

- The input signal $x_a(t)$ will be properly sampled and analyzed if $F_s \geq 2 F_{max}$ or $F_{max} \leq F_s / 2$
- Principal values for discrete-time frequency

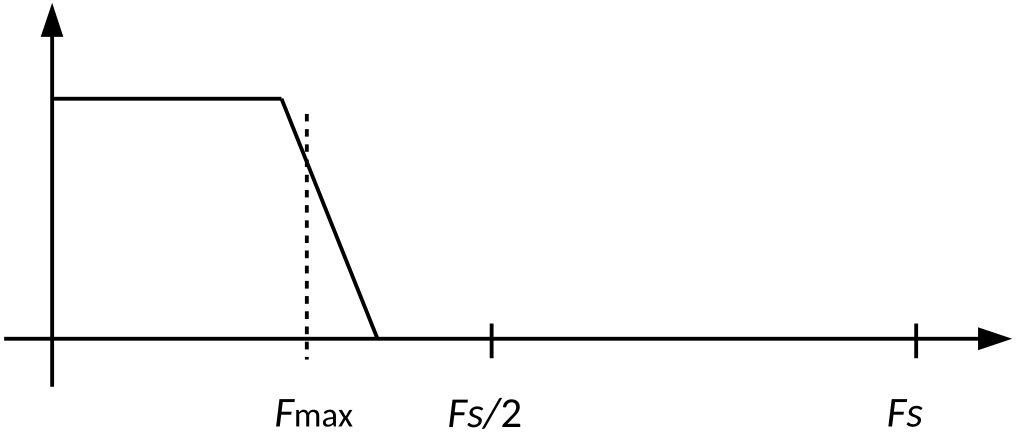
$0 \leq f \leq \frac{1}{2}$ → less than 1/2 cycles per sample, [*cyc/smp*]

$0 \leq \omega \leq \pi, \quad \omega = 2\pi f$ → less than 1/2 radians per sample, [*rad/smp*]

The Nyquist sampling theorem

- Avoid aliasing by low-pass filtering the continuous-time signal *x(t)* before sampling
- In practice sample signals at about $F_s = (3 - 4).\ F_{max}$

- Analog low-pass filter (anti-alias filter)



The Nyquist sampling theorem, units

Lowpass filtering and sampling

$$x_a(t) = A\sin(2\pi F t + \theta) \longrightarrow x_a[nT_s] = A\sin(2\pi F/F_s\, n + \theta) = x[n]$$

$$= A\sin(\Omega t + \theta) \qquad = A\sin(\omega n + \theta) = x[n]$$

$$\Omega = 2\pi F \longrightarrow \omega = 2\pi F/F_s \rightarrow \omega = 2\pi f$$

(F_s = 6 kHz)

Ω , $-\infty < \Omega < \infty$, $\Omega = 2\pi F$,

the frequency in radians per sec [*rad/s*]

F , $-\infty < F < \infty$,

the frequency in cycles per sec or Hertz [*Hz*]

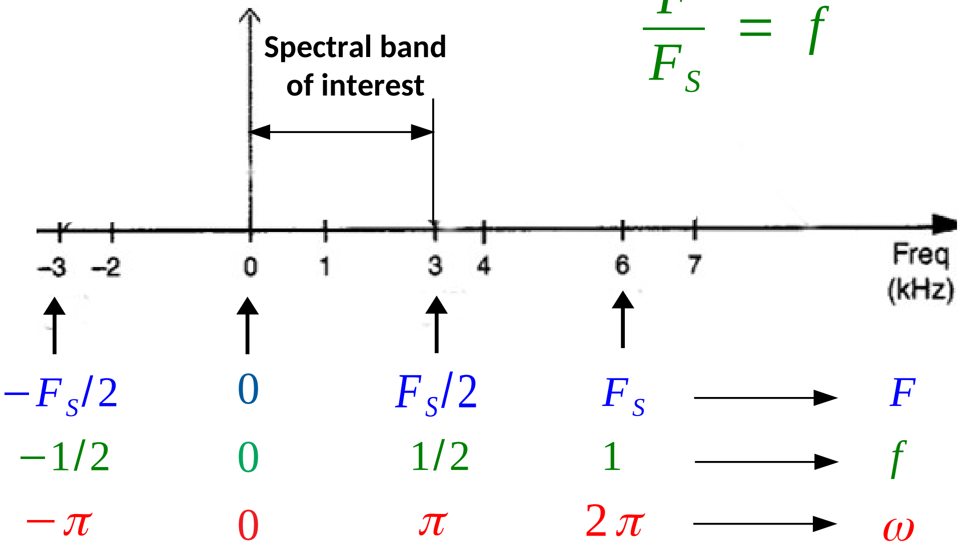
f , $-1/2 \le f \le 1/2$,

the frequency in cycles per sample [*cyc/smp*]

ω , $-\pi \le \omega \le \pi$, $\omega = 2\pi f$,

the frequency in radians per sample [*rad/smp*]

$$\frac{F}{F_S} = f$$



$-F_S/2$	0	$F_S/2$	F_S	$\longrightarrow$	F
$-1/2$	0	$1/2$	1	$\longrightarrow$	f
$-\pi$	0	π	2π	$\longrightarrow$	ω

The Nyquist sampling theorem

- Example of exam task:

  - A continuous signal is composed from three components (pure tones)

- $x_a(t) = 2\cos(2000\pi t) + 4\sin(6000\pi t) + 8\cos(12000\pi t)$

Determine the lowest sampling frequency *Fs* necessary to avoid aliasing

- *Demo on sampling, aliasing and Nyquist theorem*

  https://www.youtube.com/watch?v=yWqrx08UeUs

The Nyquist sampling theorem

- Example of exam task:
  - A continuous-time signal $x_a(t) = 4\,sin(1200\,\pi\,t)$ is sampled by the sampling frequency of $F_s = 2400$ Hz. Write the expression for such obtained discrete-time signal $x(n)$. Determine the discrete-time frequency of $x(n)$. Write also the first six samples of discrete-time signal $x(n)$.

Representation of discrete signals

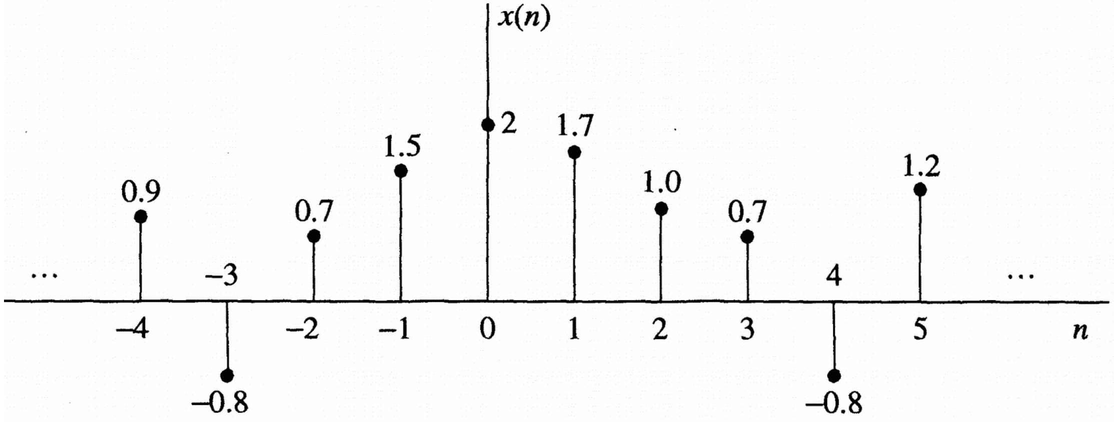
- Functional representation

$$x(n)=\begin{cases}1, & \text{for } n=1,3\\ 4, & \text{for } n=2\\ 0, & \text{elsewhere}\end{cases}$$

- Tabular representation

n	... -2 -1 0 1 2 3 4 5 ...
x(n)	... 0 0 0 1 4 1 0 0 ...

- Graphical representation



Representation of discrete signals

- Sequence representation $x(n) = \{2,\ 1.7,\ \underset{\uparrow\ n=0}{2},\ 1.5,\ 1,\ 1.8,\ \dots\}$

- Sequence representation $x(n) = \{2,\ 1.7,\ \mathbf{2},\ 1.5,\ 1,\ 1.8,\ \dots\}$

- Left- and right-sided discrete signals
  - $x(n)$ may be defined only for certain n
    - * $N_1 \leq n \leq N_2$ - Finite length (Length $= N_2 - N_1 + 1$)
    - * $N_1 \leq n$ - Right-sided
    - * $n \leq N_2$ - Left-sided
  - $x(n)$ can always be extended with zero-padding

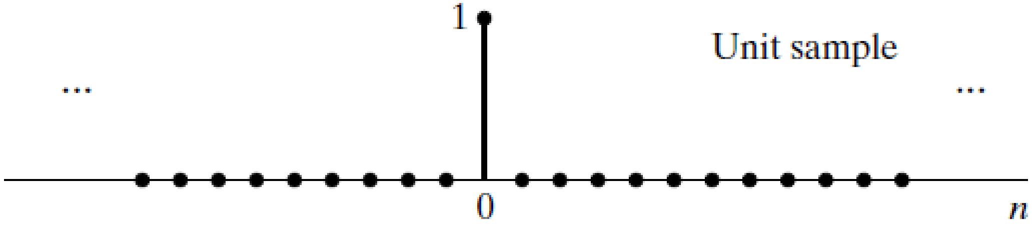
Basic signals

- Basic signals
  - The unit sample signal
  - The unit step signal
  - The rectangular signal
  - The exponential signal
  - The sinusoidal signal
- All other signals can be represented by the basic signals above
  - Fourier series
  - Fourier transformation

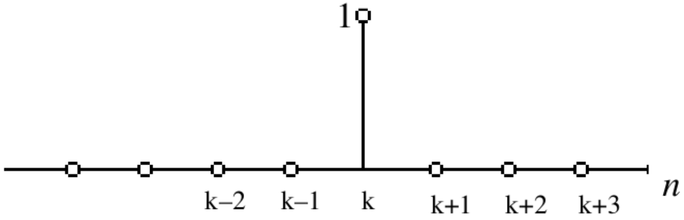
Basic signals

- The unit sample signal, *δ(n)*, unit sample sequence, unit impulse, unit function, impulse function, delta function

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



- Shift in time, *δ(n – k)*



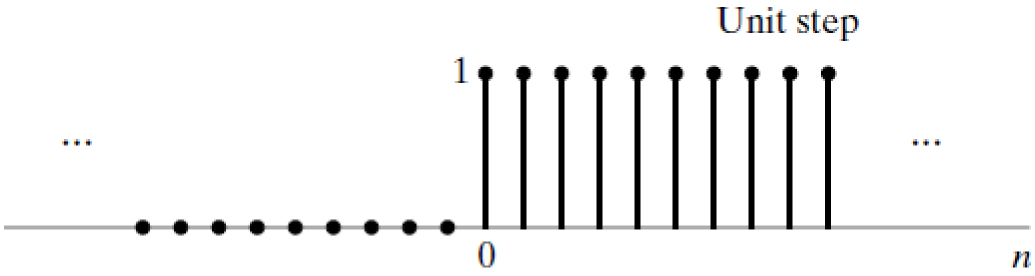
- Can express any signal *x(n)* with *δ(n)*

$$x(n) = \{\underset{\uparrow\, n=0}{A_0}, A_1, A_2, \dots\} = A_0.\delta(n) + A_1.\delta(n-1) + A_2.\delta(n-2) + \dots$$

Basic signals

- The unit step signal, *u(n)*, unit step sequence, unit step function

$$u(n)=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}$$



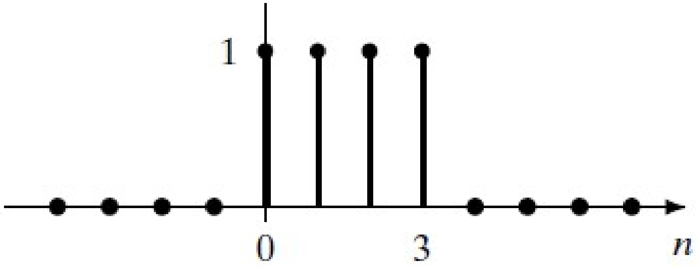
- Relate to unit sample

$$\delta(n) = u(n) - u(n-1)$$

$$u(n) = \sum_{k=0}^{n} \delta(k)$$

Basic signals

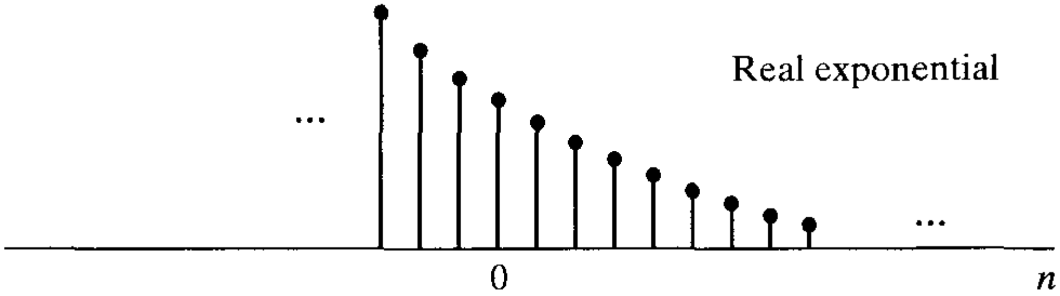
- The rectangular signal, *w(n)*, rectangular window, rectangular function ($N = 4$)



$$w(n) = u(n) - u(n-N) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

Basic signals

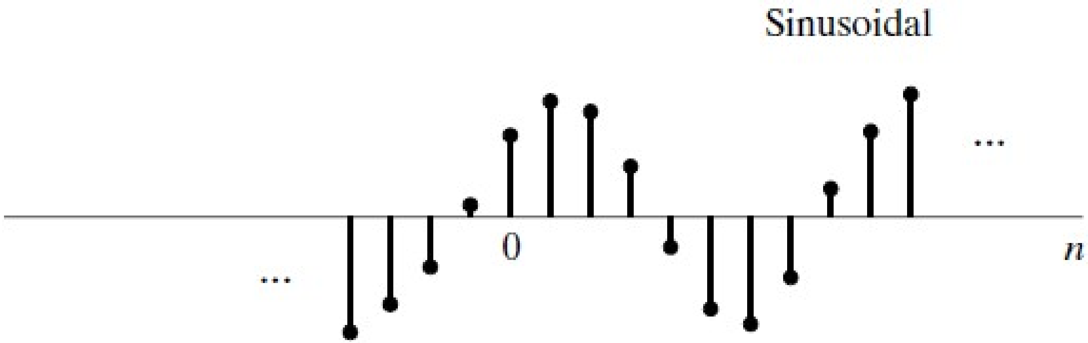
- The exponential signal, *x*(*n*), exponential sequence



$$x(n) = A.\alpha^n \qquad A > 0, \quad 0 < \alpha < 1$$

Basic signals

- The sinusoidal signal, *x(n)*



$$x(n) = A \sin\left(2\pi \frac{F}{F_s} n + \theta\right) = A \sin\left(2\pi f n + \theta\right)$$

Properties of discrete signals

- Periodic signals and aperiodic signals

  - A signal $x(n)$ is periodic with period N $(N > 0)$ if and only if

$$x(n + N) = x(n) \quad \text{for } \forall\, n$$

- Symmetric (even) and antisymmetric (odd) signals

  - A real-valued signal $x(n)$ is symmetric (even) if

$$x(-n) = x(n) \quad \text{for } \forall\, n$$

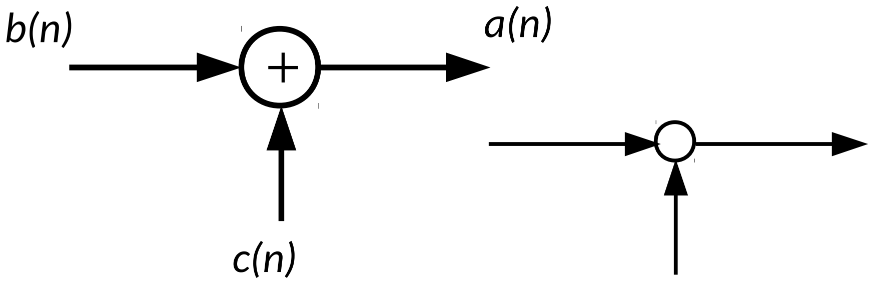
  - A real-valued signal x(n) is antisymmetric (odd) if

$$x(-n) = -x(n) \quad \text{for } \forall\, n$$

Operations on discrete signals

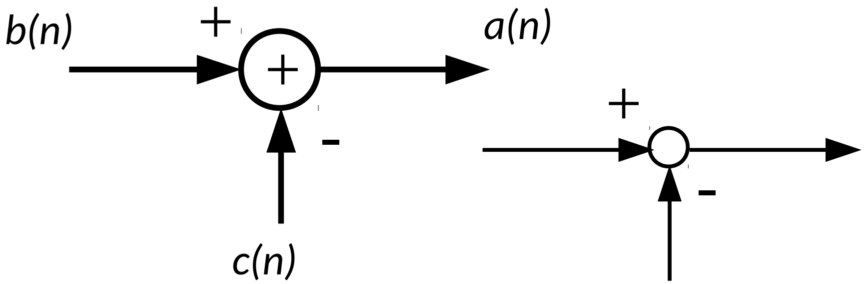
- Addition operation, adder,

  $a(n) = b(n) + c(n)$



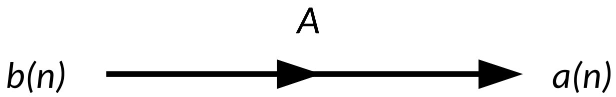
- Subtraction operation, subtractor,

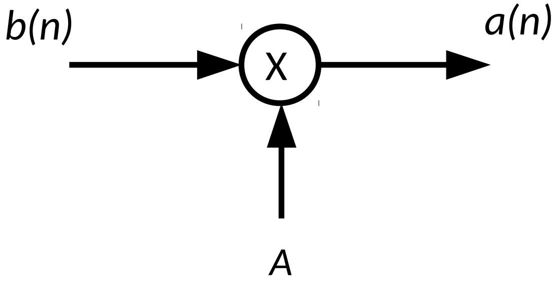
  $a(n) = b(n) - c(n)$



- Multiplication with a constant,

  $a(n) = A . b(n)$

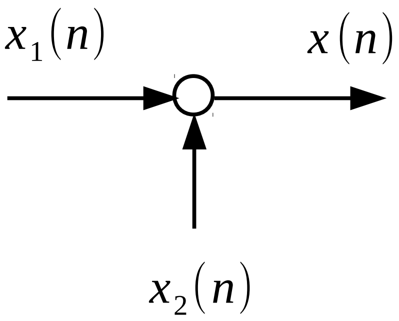


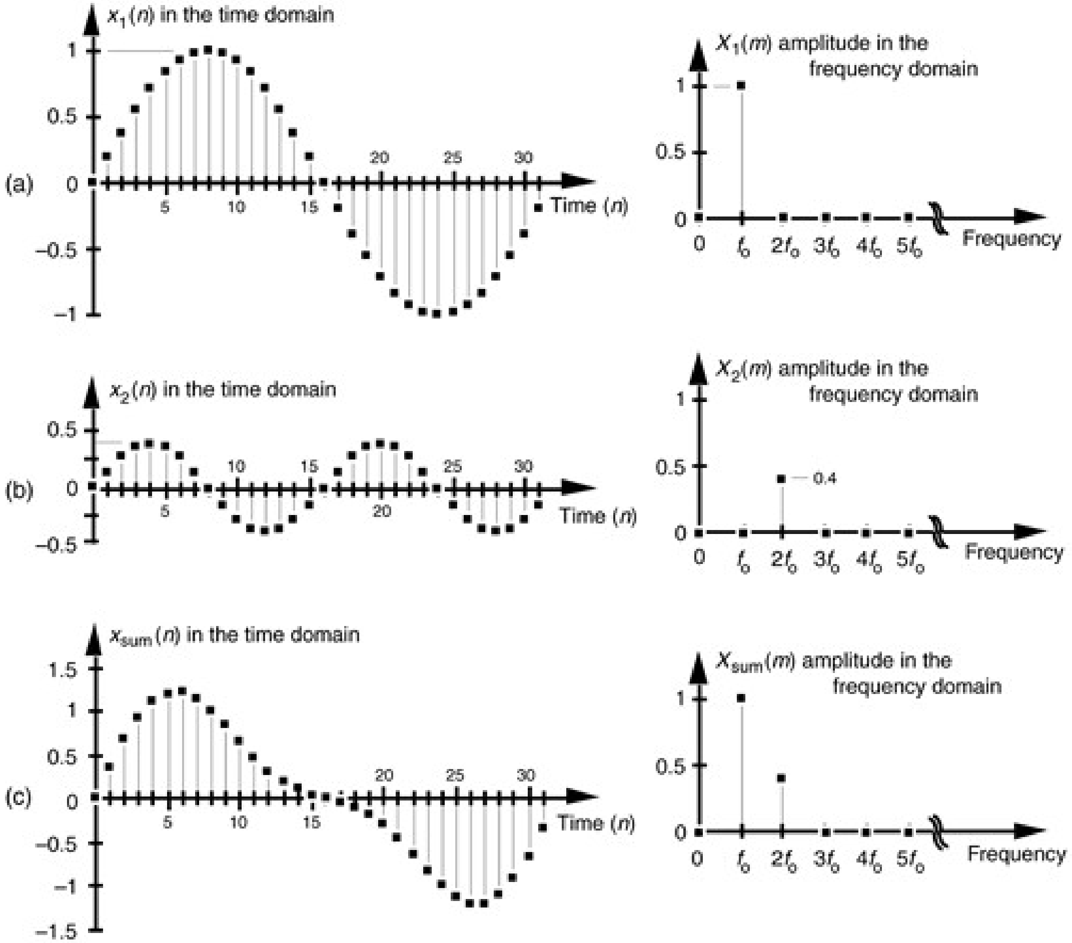


Operations on discrete signals

$$x(n) = x_1(n) + x_2(n) = \sin(2\pi F_0 T_s n) + 0.4.\sin(2\pi 2 F_0 T_s n)$$

- Summing sinusoids





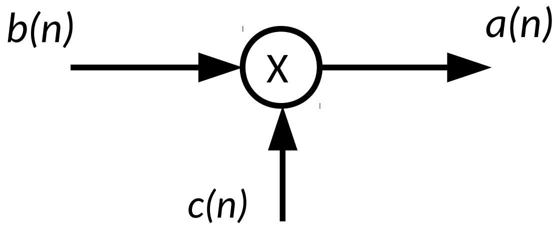
[Lyons]

Operations on discrete signals

- A signal multiplier, e.g., windowing

  $a(n) = b(n)\,c(n) = b(n) \,.\, c(n)$

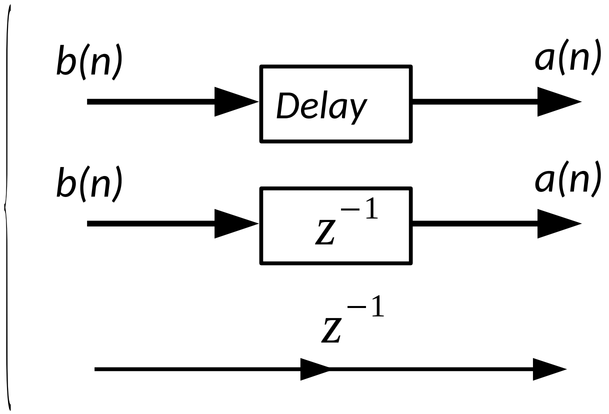
  $= b(n) \times c(n)$



- Time shifting, $a(n) = b(n - N)$

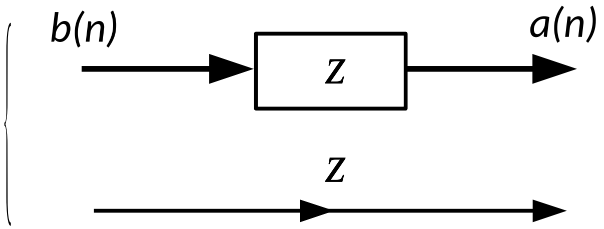
  - If $N > 0$, delaying operation,

    a unit delay, $a(n) = b(n - 1)$



  - If $N < 0$, advancing operation,

    a unit advance, $a(n) = b(n + 1)$



Combination of basic operations

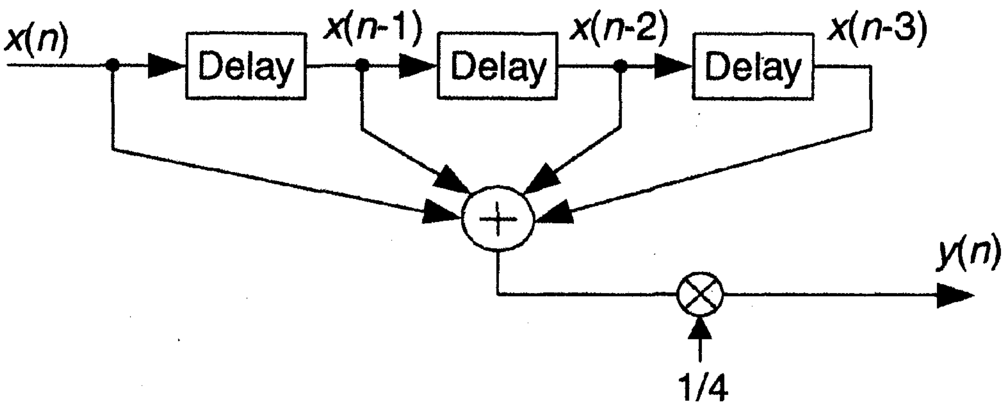
- Example $$y(n) = \frac{1}{4}[x(n)+x(n-1)+x(n-2)+x(n-3)] = \frac{1}{4}\sum_{k=n-3}^{n} x(k)$$

- What is block diagram realization of difference equation?

Combination of basic operations

- Example $y(n) = \frac{1}{4}[x(n)+x(n-1)+x(n-2)+x(n-3)] = \frac{1}{4}\sum_{k=n-3}^{n} x(k)$

- Block diagram realization of difference equation



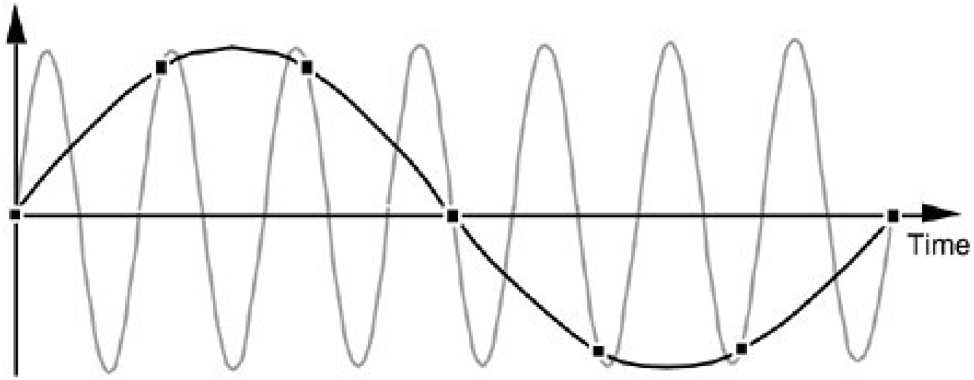
[Lyons]

(Additional materials)

- Sampling a sinusoid
- Analog to Digital (A/D) and Digital to Analog (D/A) conversion

Sampling a sinusoid

- Sampling a sinusoid is ambiguous



$$x_1(n) = \sin(\omega_0 n)$$

$$x_2(n) = \sin((\omega_0 + 2\pi r)n) = \sin(\omega_0 n) = x_1(n)$$

For $\sin(\omega n)$, $\omega = 2\pi r \pm \omega_0$

all r (integers) appears the same after sampling → aliasing

Analog to Digital (A/D) and Digital to Analog (D/A) conversion

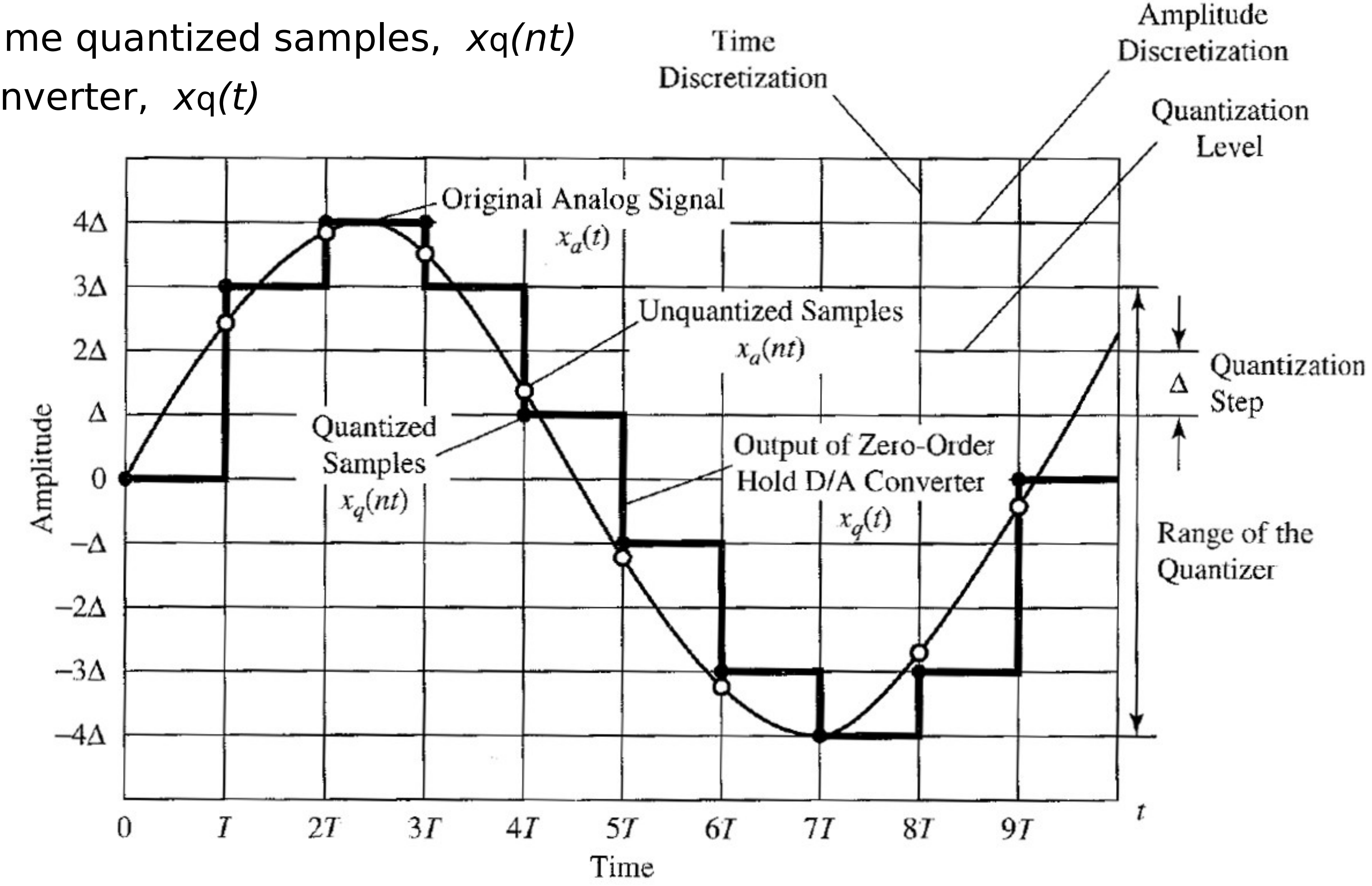
- Continuous-time signal, $x_a(t)$
- Discrete-time unquantized samples, $x_a(nt)$
- A/D → Discrete-time quantized samples, $x_q(nt)$
- Output of D/A converter, $x_q(t)$

$x_a(nT) \rightarrow x(n)$

Discrete signal

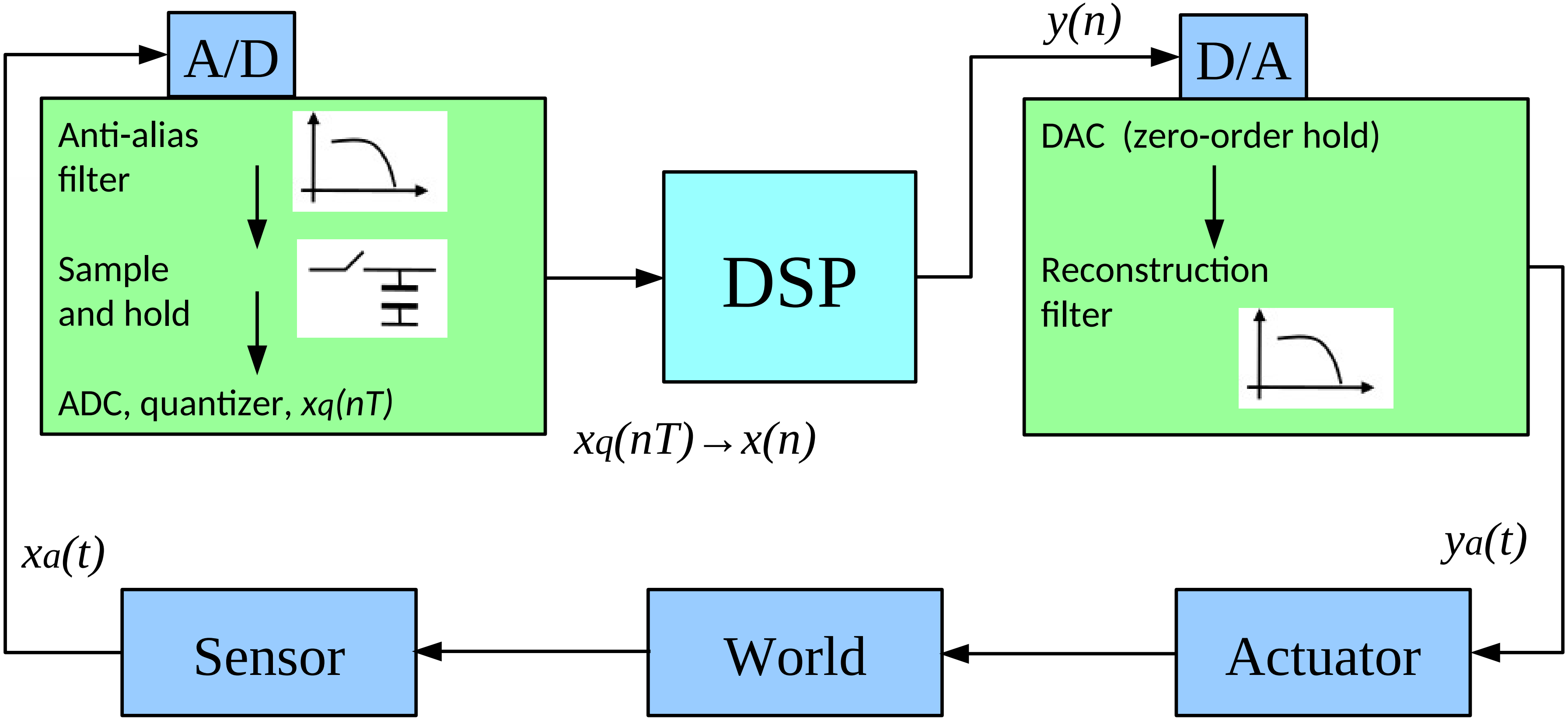
$x_q(nT) \rightarrow x(n)$

Digital signal



[Proakis, Manolakis]

Analog to Digital (A/D) and Digital to Analog (D/A) conversion



A/D

Anti-alias filter

Sample and hold

ADC, quantizer, $xq(nT)$

DSP

$xq(nT) \rightarrow x(n)$

$y(n)$

D/A

DAC (zero-order hold)

Reconstruction filter

$ya(t)$

Actuator

World

Sensor

$xa(t)$

- DSP must interact with an analog world